Geometric approaches to state feedback control for continuous and switched linear systems

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ABSTRACT

The usability of underdetermined state feedback in the context of pole placement, eigenstructure assignment, stabilization and robustness analysis of continuous and switched linear systems is explored. Analytical tools for system design and analysis are discussed from the geometric perspective. Thereby, several new results are obtained and some well-known ones are recovered and generalized by utilization of the computational methods based on the concept of invariance subspaces.

Key Words: Pole placement, eigenstructure assignment, invariant subspaces, robustness, switched linear systems.

I. Introduction

Motivation and Outlook: It is well-known that for a given set of self-conjugate distinct eigenvalues \( \{ \lambda_i \in \mathbb{C}; \ i \in r \} \) and a corresponding set of self-conjugate linearly independent right eigenvectors \( \{ \nu_i \in \mathbb{C}^n; \ i \in r \} \), \( r \leq n \), the right partial eigenstructure assignment problem, consisting in solving for \( K \in \mathbb{R}^{n \times m} \) in \((A + BK^T)\nu_i = \lambda_i \nu_i, \ i \in r\), has always a solution provided that \((A, B)\) is controllable and the eigenvectors \( \nu_i \) lie in \( \text{Range}(N_i) \), where

\[
[A - \lambda_i I, B] \begin{bmatrix} N_i \\ M_i \end{bmatrix} = 0,
\]

\(N_i \in \mathbb{C}^{n \times r}\), and \(M_i \in \mathbb{C}^{m \times r}\), [13]. The state-feedback gain \( K^T \) is then given by the solution of the equation

\[
K^T[\nu_1, \ldots, \nu_r] = [m_1, \ldots, m_r],
\]

with \( m_i \in \text{Range}(M_i) \) for \( i \in r \), which is clearly under-determined for \( r < n \). It has a unique solution for \( r = n \), which refers to the complete eigenstructure assignment. On the other hand, the left partial eigenstructure assignment problem, requiring \( \omega_i^T(A + BK^T) = \lambda_i \omega_i^T \) for \( i \in r \), where \( \omega_i \in \mathbb{C}^n \) are arbitrary self-conjugate linearly independent, leads to

\[
[\lambda_i^* I - A^T, I] \begin{bmatrix} F_i \\ G_i \end{bmatrix} = 0, \tag{1a}
\]

and

\[
KB^T[\omega_1, \ldots, \omega_r] = [g_1, \ldots, g_r], \tag{1b}
\]

where \( \omega_i \in \text{Range}(F_i) = \mathbb{C}^n \) and \( g_i \in \text{Range}(G_i) \). As \( F_i, G_i \in \mathbb{C}^{n \times r} \), and \( B_i \in \mathbb{R}^{n \times m} \), Eq. (1a) is overdetermined with respect to \( K \) if \( r > m \), otherwise (for \( r \leq m \)) a solution for \( K \) always exists. Hence, in addition to distinct roles in the design of a control system [see, e.g., 3], right and left eigenstructure assignment problems are technically different. The feasibility in the former approach is subject to desired eigenvectors, and in the latter one, to the availability of sufficient degrees of freedom at the system input. In fact, nearly all eigenstructure assignment problems are motivated either by the non-feasible desired right eigenvectors or/and by the lack of the degrees of freedom at the input. Moreover, they typically involve complete (rather than partial) pole placement. Being, thus, predominant in either approach, the overdetermined problem in \( K \) has given rise to a vast number of problem formulations and algorithms, including [18, 5, 3, 16, 4, 12].

In this article, a rather dual approach is followed: we consider the under-determined case \( m \geq r \) in (1a) and derive analytic conditions in diverse feedback control problems, including pole placement, partial...
eigenstructure assignment and stabilization of switched linear systems. The discussion of the first problem for single-input systems leads us to a novel generalized pole-placement formula in Section II, which to our best knowledge, has not appeared in that form previously in the literature. However, we do stress its close relationship to the sequential pole-placement algorithm of Simon and Mitter in [17], and also establish a direct link to the well-known Bass-Gura and Ackermann formulae. To avoid numerical problems related to large-order matrix inversion, we extend the idea to a sequential algorithm, whereby specific invariant subspaces need to be computed. It turns out, that this represents a generalization of the so-called method of invariant planes [1]. For the case of multi-input systems with \( m = r + 1 \), in Section III, we pose a constructive algorithm for partial eigenstructure and complete eigenvalue assignment. We establish conditions on partial left eigenstructure assignment that preserve the right-invariance of invariant subspaces in closed loop systems, which are utilized in the context of stabilization of a class of switched systems. In Section IV, we apply these tools for stabilization of a class of switched linear systems which assumes a common right invariant subspace shared by the constituents. We follow a geometric approach by composing, in the first step, a common quadratic Lyapunov function (CQLF) out of sub-CQLFs of the constituent subsystems corresponding to the common invariant subspace and its complementary one. While such class of systems is quite restricted, in a further step, we establish methods for characterizing the robustness margins of the composed CQLF with respect to perturbations of the canonical angles of the underlying invariant subspace(s), this giving rise to an insight into the robustness of the common quadratic stability. In this manner, we relax the stringent requirement for a common invariant subspace. Thereby, we gain an insight into the impact of the geometry of the system to its stability. Additionally, by making use of the left eigenstructure constructions of the previous section, we suggest high-gain regularization procedures to guarantee some constraints concerning the constituent systems. Finally, we extend the ideas to complete invariant chains and derive some new proofs to well-known results concerning commutative and simultaneously triangulizable switching constituents. We further deduce the fact that a switching system with pairwise rank one commutators does share a diagonal Lyapunov function. For illustration purposes, we present several simple numerical examples.

For the sake of completeness we present in the sequel notation and some basic prerequisites from linear algebra and systems theory.

**Notation:** \( \mathbb{R} \) and \( \mathbb{C} \) are the fields of real and complex numbers, respectively. \( \mathbb{C}^+ \) stands for the open left-hand complex half-plane. We designate the set of all \( m \times n \) matrices over \( \mathbb{R} \) (or \( \mathbb{C} \)) by \( \mathbb{R}^{m \times n} \) (or \( \mathbb{C}^{m \times n} \)), and denote the conjugate transpose of a matrix \( M \) by \( M^H = (M)^T \). The range of a matrix \( V \in \mathbb{C}^{n \times r} \) is denoted by \( \text{Range}(V) \); the null-space by \( \text{Kern}(V) \). The pseudo-inverse of \( M \), defined by \( M^+ := (M^HM)^{-1}M^H \) if the columns of \( M \) are linearly independent. If the rows of \( M \) are linearly independent, then \( M^+ := M^H(MM^H)^{-1} \). A square matrix \( M \) is Hermitian (symmetric) if \( M = M^H \). It is unitary if \( M^H = M^{-1} \). A Hermitian matrix is positive (negative) definite if all its eigenvalues are real positive (negative), which we denote by \( M > 0 \) (\( M < 0 \)). If \( M > 0 \) then there exists a unique matrix \( L > 0 \) such that \( L^2 = M \). \( L \) is called the square root of \( M \) and is denoted by \( M^{1/2} \).

Let \( M \) be partitioned as:

\[
M = \begin{pmatrix}
  A & B \\
  C & D
\end{pmatrix}
\]  

(2)

If \( A \) is nonsingular, then its Schur complement in \( M \) is defined by \( M/A := D - CA^{-1}B \) [see, e.g., 21]. Then, \( M < 0 \) if and only if \( A < 0 \) and \( M/A < 0 \). The inertia of a Hermitian matrix is defined by the triplet \( \text{In}(M) = (\pi(M), \nu(M), \delta(M)) \), where \( \pi(M), \nu(M), \delta(M) \) are the numbers of the positive, negative and zero eigenvalues of \( M \) counted with multiplicities, respectively. The spectral norm of \( M \) is defined as \( \|M\| := \sup_{\|x\|=1} \|Mx\| \). It is equal to the largest singular value of \( M \), \( \|M\| = \sigma_{\text{max}}(M) \). Hence, \( \|M^H M\| = \|M\|^2 \). Finally, note that we use often times the shorthand: \( r := \{1, \ldots, r\} \) for any \( r \in \mathbb{N} \), which is mostly employed in the context of the inclusion \( i \in \{1, \ldots, r\} \) and shorthanded by \( i \in r \).

**Systems:** We consider the open-loop linear time-invariant (LTI) system of the form:

\[
\Sigma_o: \quad \dot{x} = Ax + Bu
\]  

(3)

with \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), and \( m \leq n \). For notation simplicity, this system is sometimes denoted by the pair \([A, B]\). The corresponding closed loop system matrix is denoted by \( \bar{A} := A + BK^T \), where \( K^T \in \mathbb{R}^{m \times n} \) represents the state-feedback gain in \( u = K^T x \). The controllability matrix referring to the pair \([A, B]\) is denoted by \( C[A, B] := \]
[B, AB, ..., A^{n-1}B]. Generally, the open-loop eigenvalues of $A$ are denoted by $\mu$; the closed loop ones of $\bar{A}$ by $\lambda$. The set of eigenvalues is denoted by $\Lambda(\cdot)$. Unless otherwise stated, the input matrix $B$ is assumed to be full-rank.

**Invariant subspaces:** Let $A \in \mathbb{C}^{n \times n}$. The subspace $V \subseteq \mathbb{C}^n$ is said to be $A-$invariant if $AV \subseteq V$. $V$ is $A$-invariant if and only if there exists a matrix $Y$ such that $AV = VY$ and $Range(V) = V$. Analogously, $U \subseteq \mathbb{C}^n$ is said to be left $A$-invariant if $AU = UA$ and $Range(U) = U$. $V$ and $U$ are said to be complementary if $U \oplus V = \mathbb{C}^n$, where $\oplus$ stands for the direct sum. These definitions and facts are conveniently described in the operator form by:

$$A = \begin{bmatrix} X & 0 \\ * & Y \end{bmatrix} : \oplus \rightarrow \oplus. \quad (4)$$

This description introduces an invertible matrix $T = [U, V]$ such that $A = T \Sigma T^{-1}$, where $\Sigma$, as defined above, is lower block-triangular. Hereby, $U$ and $V$ are not necessarily perpendicular, but for computation convenience, we will often let $U \perp V$. In this case, we can chose a unitary $T$ with $U^H A = I$, $V^H V = I$, $U^H V = 0$, $V^H U = 0$. $X$ and $Y$ refer to the restricted maps of $A$ to $U$ and $V$, respectively, i.e. $X = A|_U$ and $Y = A|_V$. The zero block in this picture indicates that all nonzero elements of $V$ are mapped to $V$ via $A$ (i.e. their $U$-component is 0). $\mathbb{C}^n$ and $\emptyset$ are trivial invariant subspaces. A non-trivial $A$-invariant subspace is the eigenspace of an eigenvalue, such as the span of a corresponding eigenvector $v$, because of $Av = \lambda v$. In other words, whatever $A$, there exists always a one-dimensional $A$-invariant subspace. Clearly, left eigenvectors $\omega$ constitute left invariant subspaces, as $\omega^H A = \lambda \omega^H$. If $\nu_0, \nu_1, ..., \nu_k$ is Jordan chain of $A$ corresponding to an eigenvalue $\lambda$ (i.e. $\nu_0$ is an eigenvector, and the rest are generalized eigenvectors), then $\text{Span}(\nu_0, \nu_1, ..., \nu_k)$ is $A$-invariant. Consider the (right) Schur decomposition of $A$: $AQ = QY$, where $Q$ is a unitary matrix and $Y$ is upper- (or, lower-) triangular. [Note that in the real case, that is if $Q$ is real orthogonal, then $Y$ is block upper- (lower-) triangular, with $1 \times 1$ or $2 \times 2$ along the diagonal.] Then, the Schur vectors, i.e. the columns of $Q$ in the upper-triangular form, $q_1, ..., q_n$, constitute a chain of invariant subspaces $\emptyset \subseteq M_1 \subseteq M_2 \subseteq ... \subseteq M_n = \mathbb{C}^n$, where $M_i = \text{Span}(q_1, ..., q_i)$. If $\dim(M_i) = i$, then the chain is said to be complete.

**Projectors and canonical angles:** A matrix $P \in \mathbb{C}^{n \times n}$ is said to be projector if it is idempotent, i.e. $P^2 = P$. The important feature of projectors is that there exists a one-to-one correspondence between the set of all projectors and the set of all complementary subspaces in $\mathbb{C}^n$, [7]. $P$ is said to be an oblique projector onto $V$ along $U$ if $Range(P) = V$ and $Kernel(P) = U$. The expression for the oblique projector is given by $P = V(V^H V)^{-1}V^H$, where $U$ spans $U^\perp$. $Q = I - P$ is a projector and it is called complementary projector to $P$. $P$ is said to be an orthogonal projector onto $V$, usually denoted by $P_V$, if $U = V^\perp$, that is, if $U^H V = 0$. Then $P_V = V(V^H V)^{-1}V^H$. A projector $P$ is orthogonal if and only if it is Hermitian. Note that $\|P\| \geq 1$, unless it is orthogonal, yielding $\|P\| = 1$.

Another important feature is that projectors provide simple tools to characterize the gap (or, angle) between two invariant subspaces. The gap between two subspaces can be characterized in terms of angles. The minimal angle $\theta_{\text{min}}$ between two arbitrary (i.e. not necessarily complementary) closed subspaces $U$ and $V$ is defined as [10], [19]:

$$\cos \theta_{\text{min}}(U, V) := \sup_{u \in U, v \in V} \frac{|v^H u|}{\|u\|_1 \|v\|_1}. \quad (5)$$

It turns out that the minimal angles can be expressed in terms of the norms of orthogonal projectors as: $\cos \theta_{\text{min}}(U, V) = \|P_U P_V\|$, see [19], [10]. Often times the gap (or maximal angle $\theta_{\text{max}}$) is rather useful:

$$\cos \theta_{\text{max}}(U, V) := \inf_{u \in U, v \in V} \frac{|v^H u|}{\|u\|_1 \|v\|_1}. \quad (6)$$

If $U$ and $V$ are complementary, then the following useful relations arise:

$$\sin \theta_{\text{min}}(U, V) = \frac{1}{\|P\|}, \quad \text{and} \quad (7)$$

$$\theta_{\text{min}}(U, V) + \theta_{\text{max}}(U, V^\perp) = \frac{\pi}{2} \quad (8)$$

**II. Spectrum assignment**

Consider the state space representation of a finite-dimensional controllable single-input linear time invariant system: $\dot{x} = Ax + Bu$. It is well-known that for any arbitrary multiset of self-conjugate eigenvalues in $\{\lambda_i\} \subseteq \mathbb{C}$, there exists always a unique state feedback gain $k \in \mathbb{R}^n$ which solves the pole assignment problem [11]. In the sequel, we provide an original method for computation of $k$. 

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Let $\omega_1 \in \mathbb{C}^n$ be a left eigenvector of the closed loop system matrix $A := A + bk^T$ corresponding to an arbitrary eigenvalue $\lambda_1 \in \mathbb{C}$. Then, with $\omega^H_{n-1}(A + bk^T) = \lambda_1 \omega^H_{n-1}$, we claim:

$$k^T = \omega^H_{n-1}(\lambda_1 I - A) \quad \text{and} \quad \omega^H_{n-1} b = 1,$$  

(9)

whereby in light of the implementation, care has to be taken in selecting a pair $\omega_{n-1}$ and $\lambda_1$ that guarantees a real outcome $k \in \mathbb{R}^n$. Observe, that the right-hand side statement in (9) results from the fact that $\omega_{n-1}/\omega^H_{n-1} b$ is a left eigenvector of $A$, as well, and the condition $\omega^H_{n-1} b \neq 0$ which is guaranteed by the controllability of the pair $(A, b)$. Indeed, if the opposite would hold true, i.e. if $\omega^H_{n-1} b = 0$, we would have: $\omega^H_{n-1}(A + bk^T) = \omega^H_{n-1}A = \lambda_1 \omega^H_{n-1}$ for all $k$, indicating that $\lambda_1$ is an eigenvalue of $A$ and $\tilde{A}$ simultaneously, i.e. it cannot be shifted by any $k$, which contradicts the controllability of $(A, b)$.

Furthermore, Eq. (9) reveals that the remaining eigenvalues in the multiset $\{\lambda_j\}_{j=2}^n$ are uniquely specified by the left eigenvector $\omega_{n-1}$. Hence, it is natural to pose the spectrum assignment in terms of computation of the eigenvector $\omega_{n-1}$ such that a prespecified multiset of self-conjugate (not necessarily distinct) eigenvalues $\{\lambda_j\}_{j=2}^n$ are assigned to

$$\tilde{A} = (I - b\omega^H_{n-1})A + \lambda_1 b\omega^H_{n-1}.$$  

(10)

To this end, we start with the characteristic polynomial of the closed loop matrix $A$, which (with a little of technical effort) is shown to be given by:

$$\det (\lambda I - \tilde{A}) = (\lambda - \lambda_1)\omega^H_{n-1}\text{adj}(\lambda I - A^T)b.$$  

(11)

Next, consider the controller canonical form $\xi = A_c\xi + b_c u$, with $TA_c = AT$, $Tb_c = b$, and

$$A_c = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{pmatrix}, \quad b_c = \begin{pmatrix} 0 \\ \vdots \\ \vdots \end{pmatrix}.$$  

Here, $T := CC_0^{-1}$ [11] indicates the transformation $x = T\xi$, where, for convenience, we denote by $C := C[A, b]$ and $C_c := C[A_c, b_c]$ the open-loop and closed loop controllability matrix [11]. The characteristic polynomial of $\tilde{A}$ then reads:

$$p(\lambda) = \det (\lambda I - \tilde{A}) = \lambda^n + a_1 \lambda^{n-1} + \ldots + a_n.$$  

(12)

Following the discussion related to Eq. (9), if we let

$$\gamma^H_{n-1} = [\gamma_{n-1,n-1}, \ldots, \gamma_{n-1,1}, 1]$$  

(13)

represent the desired left eigenvector, and $\lambda_1$ the corresponding eigenvalue of the closed loop $\tilde{A}_c := A_c + b_c k^T = T^{-1}AT$ in the $\xi$-coordinates, then from (11) we get:

$$\det (\lambda I - \tilde{A}_c) = (\lambda - \lambda_1)\gamma^H_{n-1} \Upsilon(\lambda),$$  

(14)

where we introduce: $\Upsilon(\lambda) := [1 \lambda \ldots \lambda^{n-1}]^T = \text{adj}(\lambda I - A^T)b$. From (14) it is obvious that the eigenvalues $\{\lambda_j\}_{j=2}^n$ of the closed loop matrix $\tilde{A}_c$ (that is, of $\bar{A}$, as well) are independent of the parameters $a_1, \ldots, a_n$, and they are entirely determined by the left eigenvector $\gamma_{n-1}$. On the other hand, let (14) be specified by a desired closed loop characteristic polynomial of the form:

$$q_n(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \ldots + a_n.$$  

(15)

Eq. (14) reveals that $\gamma_{n-1}$ hosts the parameters of the polynomial $q_n(\lambda)$, where $q_n(\lambda) = (\lambda - \lambda_1)q_{n-1}(\lambda)$. It can be checked that $\gamma_{n-1}$ is given by the recursive algorithm:

$$\gamma_{n-1,i} = a_i + \gamma_{n-1,i-1} \lambda_1$$  

(19)

for $i \in [n - 1]$, whereas, in accordance with the adoption in (13): $\gamma_{n-1,0} = 1$. Since $\tilde{A}$ and $\tilde{A}_c$ are similar, we further have:

$$\gamma^H_{n-1} = \gamma^H_{n-1} C_c C^{-1}, \quad k^T = \omega^H_{n-1}(\lambda_1 I - A).$$  

(16)

This represents our initial pole assignment formula. Next, we generalize it and demonstrate its relationship to the Bass-Gura and Ackermann formulae. First, it is readily verified that

$$\gamma^H_{n-1}(\lambda_1 I - A_c) = [a_1 - a_1, \ldots, a_n - a_n] := \gamma^H_{n-1},$$  

(17)

indicating that all the closed loop eigenvalues in $\{\lambda_j\}_{j=1}^n$ are “encoded” in the (real) vector $\gamma_n$, whereas $\gamma_{n-1}$ carries the information about $\{\lambda_j\}_{j=2}^n$. Then, the Bass-Gura formula:

$$k^T = \gamma^H_{n-1} C_c C^{-1}$$  

(18)

results immediately, if we rewrite (16) as: $k^T = \gamma^H_{n-1}(\lambda_1 I - A_c) C_c C^{-1}$, with the term $T^{-1} = C_c C^{-1}$ shifted right most.

Eq. (17) can be interpreted as “pulling out” or “carrying over” the eigenvalue $\lambda_1$ from $\gamma_n$ via the factor $\lambda_1 I - A_c$, this necessarily introducing $\gamma_{n-1}$. By proceeding in the same manner, we can pullout the eigenvalue $\lambda_2$ from $\gamma_{n-1}$ using $\lambda_2 I - A_c$; $\lambda_3$ from $\gamma_{n-2}$ via $\lambda_3 I - A_c$; and so on. Hence, one can introduce:

$$\gamma^H_{n-r} := [\gamma_{n-r,n-1}, \ldots, \gamma_{n-r,1}, 0, \ldots, 0],$$  

(19)

where $r \in n_0$ using:

$$\gamma^H_{n} = \gamma^H_{n-r} \prod_{i=1}^r (\lambda_i I - A_c),$$  

(20)
where the \((r - 1)\)–zeros (for \(r \geq 2\)) result due to the “absence” of the eigenvalues \(\lambda_2, \ldots, \lambda_r\) in \(\gamma_{n-r}\), while the \(n - r\) non-zero terms carry the information about \(\lambda_{r+1}, \ldots, \lambda_n\). In this sense, by substituting \((20)\) into \((18)\), our spectrum assignment formula \((16)\) can be set in the general form:

\[
k^T = \gamma_{n-r}^{H} C_c C^{-1} \prod_{i=1}^{r} (\lambda_i I - A), \quad r \in n, \tag{21}
\]

which can be slightly generalized to

\[
k^T = \omega_{n-r}^{H} q_r(A), \quad r \in n_0, \tag{22}
\]

with \(q_0(A) := I_n\) and otherwise:

\[
\omega_{n-r}^{H} := \gamma_{n-r}^{H} C_c C^{-1}, \quad q_r(A) := \prod_{i=1}^{r} (\lambda_i I - A). \tag{23}
\]

Clearly, equation \((22)\) represents the generalized form of our initial expression in \((16)\). For \(r \geq 1\) the vector \(\gamma_{n-r}\) is simply defined by the coefficients of the polynomial \(q_{n-r}(\lambda)\), where

\[
q_n(\lambda) = q_{n-r}(\lambda) q_r(\lambda). \tag{24}
\]

The definition of \(q_n\) [reflecting the Bass-Gura formula with \(r = 0\), cf. \((17)\)] represents an exception to this rule.

Now, consider the special case with \(r = n\), and let \(q_n(A)\) denote the real matrix polynomial corresponding to the desired characteristic polynomial \(q_n(\lambda)\) from \((15)\). Then, using \(\gamma_0^H = [1, 0, \ldots, 0]\) from \((19)\), and: \([1, 0, \ldots, 0] \cdot C_c = [0, 0, 0, 1]\), we obtain the Ackermann formula directly from \((21)\):

\[
k^T = [0, \ldots, 0, 1] C^{-1} q_n(A). \tag{25}
\]

2.1. Comments

(i) Expression \((21)\), i.e. \((22)\), provides a direct link of the Bass-Gura and Ackermann formulae. Moreover, it represents a generalization thereof: the former one results for \(r = 0\), and the latter one for \(r = n\) in \((21)\). Note also that from Ackermann formula we immediately obtain

\[
\omega_0^H = [0, \ldots, 0, 1] C^{-1}.
\]

(ii) The desired conjugate eigenpairs should be “encoded” jointly in \((21)\), either in the real vector \(\omega_{n-r}\) or in the real matrix polynomial \(q_r(A)\) to benefit from the numerical computation with real numbers. Therefore, without loss of generality we may consider

\[
k^T = \omega_{n-r}^T q_r(A), \quad r \in n_0, \tag{26}
\]
as the general form of our spectrum assignment formula. In this sense, it is also convenient to use a real \(\lambda_1\) in \((9)\). To summarize we have the following result.

**Theorem 1** Let the monomial \(q_n(\lambda) = q_{n-r}(\lambda) q_r(\lambda), r \in n_0\), represent the desired closed loop characteristic polynomial, where \(q_{n-r}(\lambda) = [1, \ldots, \lambda^{n-r-1} \gamma_{n-r}, (\gamma_{n-r} \in \mathbb{R}^n)\) and \(q_r(\lambda)\) are real polynomials in \(\lambda\) (with leading coefficients equal to one). Then, equation \((26)\) provides the solution to the pole assignment problem with \(\omega_{n-r}^T = \gamma_{n-r}^T T^{-1}\), where \(T\) refers to the canonical transformation matrix, and \(q_r(A)\) to the matrix polynomial corresponding to \(q_r(\lambda)\).

(iii) If \(\omega_{n-1}\) in \((9)\) is selected to be the left eigenvector of the open-loop matrix \(A\) corresponding to a real eigenvalue, say \(\mu_1\), then from \((10)\) we have \(\bar{A} = A + \Delta_1 b \nu_{n-1}\), with \(\Delta_1 := \lambda_1 - \mu_1\) referring to a real shift. The remainder open-loop eigenvalues \(\{\mu_i\}_{i=2}^n\) are thereby unaltered, as for any right eigenvector \(\nu_{n-i}\) of \(A\) corresponding to the eigenvalue \(\mu_i\), we have \(\bar{A} \nu_{n-i} = \mu_i \nu_{n-i}\), \(i \in \{2, \ldots, n\}\) (as a consequence of \(\omega_{n-1}^T \nu_{n-i} = 0\)). In this case we retain:

\[
k^T = \Delta_1 \omega_{n-1}^T,
\]

which represents the well-known result of Simon and Mitter [17] (cf. pp. 338). It is important to observe in this case the geometric interpretation of the vector term \(\omega_{n-1}\) in \((9)\): it is orthogonal to the invariant subspace corresponding to the eigenvalues that remain unchanged. We discuss this more generally in the next section.

(iv) Finally, due to the presence of the factor \(C^{-1}\), which for large \(n\) is typically ill-conditioned, related well-known numerical robustness problems are inherent in the expression \((21)\). In the sequel, we discuss the avoidance of such difficulties.

2.2. Numerical example

In this section, we provide a simple illustrating numerical example for Theorem 1. Consider the open-loop system given by:

\[
A = \begin{pmatrix}
0.1 & 0 & 0.1 \\
0 & 0.5 & 0.2 \\
0.2 & 0 & 0.4
\end{pmatrix}, \quad b = \begin{pmatrix}
0.01 \\
0 \\
0.005
\end{pmatrix}.
\]

The corresponding canonical transformation matrix is easily computed to be:

\[
T^{-1} = C_c C^{-1} = 10^3 \times \begin{pmatrix}
0.1923 & 1.25 & -0.3846 \\
-0.0577 & 0.625 & 0.1154 \\
0.0173 & 0.3125 & 0.1654
\end{pmatrix}.
\]
Let the desired closed loop characteristic polynomial be
\[ q_3(\lambda) = (\lambda^2 + 0.8\lambda + 0.32)(\lambda + 0.1). \]
Hereof, we directly read \( \gamma_2^T = [0.32, 0.8, 1] \), yielding
\[ \omega_2^T = \gamma_2^T T^{-1} = 10^3 \times [0.327, 1.2125, 0.1346]. \]
Moreover, we have:
\[ q_1(A) = A + 0.1I_3 = \begin{pmatrix} 0.2 & 0 & 0.1 \\ 0 & 0.6 & 0.2 \\ 0.2 & 0 & 0.5 \end{pmatrix}, \]
this, leading to the state-feedback gain:
\[ k^T = \omega_2^T q_1(A) = [33.4648, 727.5, 313.0916]. \]
Alternatively, one could interchange the roles of the two polynomial factors in that \( q_2(\lambda) = (\lambda + 0.1)(\lambda^2 + 0.8\lambda + 0.32) \) is now considered. Then: \( \gamma_2^T = [0.1, 1, 0] \) and \( \omega_2^T = [-38.47, 750, 76.94] \), while \( q_2(A) = A^2 + 0.8A + 0.32I_3. \) The reader may check that \( k^T = \omega_2^T q_2(A) \) provides the same value for the gain as in (27).

### 2.3. Partial spectrum assignment

Next we consider the design and usability of the vector \( \omega_{n-r} \in \mathbb{R}^n \) in the context of partial spectrum assignment – and, subsequently, a sequential spectrum assignment based thereupon, – which consists in shifting a subsset of open-loop self-conjugate eigenpairs, say \( M_r = \{ \mu_i \}_{i=1}^r \), to some prescribed self-conjugate \( L_r = \{ \lambda_i \}_{i=1}^r \), while keeping the remainder \( (n-r) \)-ones of \( M_{n-r} = \{ \mu_i \}_{i=r+1}^n \) unaltered \( (r < n) \).

To this end, consider the operator description of \( A \):
\[ A = \begin{bmatrix} X & \mathcal{U} \\ * & Y \end{bmatrix} : \mathcal{U} \oplus \mathcal{V} \rightarrow \mathcal{U} \oplus \mathcal{V}, \]
 correspondingly to a real similar transformation:
\[ A(U, V) = (U, V) \begin{bmatrix} X & 0 \\ * & Y \end{bmatrix}, \]
where \( \mathcal{U} \oplus \mathcal{V} = \mathbb{R}^n \) and let \( \mathcal{U} \perp \mathcal{V} \) (i.e. \( [U, V] \in \mathbb{R}^{n \times n} \) is an orthogonal matrix), \( \mathcal{U} = \text{Range}(U) \subseteq \mathbb{R}^{r}. \)
\( \mathcal{V} = \text{Range}(V) \subseteq \mathbb{R}^{n-r} \). Let further the \( A \)-invariant subspace \( \mathcal{V} \) (i.e. \( AV = VY \)) correspond to the eigenvalues collected in \( M_{n-r} \).

Next, introducing:
\[ \omega_{n-r} = U \eta \]
in terms of \( \eta \in \mathbb{R}^r \) in (26), it can be readily checked that the block-triangular form (29) is preserved under the feedback law (26) as [see also (16)]:
\[ A(U, V) = (U, V) \begin{bmatrix} X + U^T b_\eta^T q_r(X) & 0 \\ 0 & Y \end{bmatrix}. \]
This follows if (29) and (30) are substituted in \( A = A + b_\eta^T q_r(A) \) and the orthogonality property of \( (U, V) \) is utilized. Note that due to the reappearance of \( Y \) in the diagonal, the eigenvalues in \( M_{n-r} \) remain unaltered in the closed loop state matrix \( \bar{A} \), while those from \( M_r \) change subject to the parameter \( \eta \) in the term \( X + U^T b_\eta^T q_r(X) \). The latter expression suggests using the Ackermann formula for computation of \( \eta \) in shifting the eigenvalues \( M_r \) of \( X \) to \( L_r \), which are specified by the polynomial \( q_r(\cdot) \):
\[ \eta^T = [0, \ldots, 1] C^{-1}[X, U^T b]. \]
In words, if \( \omega_{n-r} \) is fixed perpendicularly (cf. (30)) onto the invariant subspace \( \mathcal{V} \), then the corresponding open-loop eigenvalues remain unchanged if we apply the feedback of the form (26) with (30) and (32). This fact provides a simple geometric interpretation for the term \( \omega_{n-r} \) in the expression (26), which has been noticed e.g. in [16]. With reference to (28), it can be relatively easily checked by utilizing the conditions \( X = U^T A U \), cf. Eq. (29), and \( UU^T + VV^T = I_n \) stemming from the orthogonality of \( (U, V) \) that the invertibility of the controllability matrix \( C^{-1}[X, U^T b] \) in (32) is equivalent to the requirement:
\[ \text{Rank } (U^T [b, A b, \ldots, A^{r-1} b]) = r, \]
which refers to the projected subsystem \( (U^T A U, U^T b) \) of \( (A, b) \) onto the subspace \( \mathcal{U} \subseteq \mathbb{R}^r \). Moreover, it follows from the latter condition that if the original pair \( [A, b] \) is controllable – which we already have assumed –, i.e. controllability matrix \( \text{Rank } C[A, b] = n \), then the projected subsystem \( (U^T A U, U^T b) \) is also controllable.

### 2.4. Sequential spectrum assignment

Comment (iv) indicates the difficulties with the invertibility of the underlying controllability matrix, while we just saw that the latter may be reduced by the projection of the system matrix onto a subspace of a lower dimension \( r < n \). This idea shall now be utilized sequentially by the following algorithm. Let
\[ A(A) = \bigcup_{\ell=1}^l M_\ell, \quad A(\bar{A}) = \bigcup_{\ell=1}^l L_\ell, \]

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where \( M_\ell \) includes a submultiset of self-conjugate open-loop eigenvalues and \( L_\ell \) the corresponding desired self-conjugate closed loop eigenvalues. In other words, the eigenvalues in \( M_\ell \) are to be shifted to \( L_\ell \) for all \( \ell \in I \), which are specified by the polynomial:

\[
q_\ell = q_1(\lambda)q_2(\lambda) \ldots q_I(\lambda). \tag{34}
\]

Then, introduce:

\[
u_\ell = \omega^T_\ell q_\ell(\bar{A}_\ell)x + u_{\ell+1}, \quad \ell \in I \tag{35}
\]

with \( u := u_1, u_{\ell+1} := 0, \bar{A}_1 := A \), and \( \bar{A}_{\ell+1} := \bar{A}_\ell + b\omega^T_\ell q_\ell(\bar{A}_\ell) \), where:

\[
\bar{A}_\ell = \begin{bmatrix} X_\ell & 0 \\ * & Y_\ell \end{bmatrix} : U_\ell \to U_\ell, \quad V_\ell \to V_\ell \tag{36}
\]

and \( V_\ell = \text{Range}(V_\ell) \) represents the \( \bar{A}_\ell \)-invariant subspace corresponding to the eigenvalues \( \lambda(\bar{A}_\ell) \setminus M_\ell \), \( U_\ell = \text{Range}(U_\ell) \) is orthogonal to \( V_\ell \) in \( \mathbb{R}^n \),

\[
\omega_\ell = U_\ell \eta_\ell, \quad \eta_\ell^T := [0, \ldots, 1] C^{-1}[X_\ell, U_\ell^T b] \tag{37}
\]

and \( q_\ell(\cdot) \) is the characteristic polynomial corresponding to the desired eigenvalues in \( L_\ell \). Effectively, we obtain:

\[
k^T = \sum_{\ell=1}^{\ell} \omega^T_\ell q_\ell(\bar{A}_\ell). \tag{38}
\]

In words, the vector \( \omega_\ell \) is set perpendicularly to the invariant subspaces \( V_\ell \) corresponding to the unaltered eigenvalues at the \( \ell \)-th iteration, while the Ackermann formula is used to design the feedback gain \( \eta_\ell \) for the assignment of the eigenvalues \( \lambda_\ell \) in the projected subspace. This procedure is repeated sequentially. Thereby, \( \bar{A}_1 \) represents the closed loop system matrix up to the \( \ell \)-th iteration. Finally, if \( M_\ell \) includes conjugated pole pairs only, then this algorithm reduces to the method of invariant planes which is introduced in [1]. In this sense, the sequential method proposed in this section represents a generalization thereof.

III. Eigenstructure assignment

In this section, we address the eigenstructure assignment problem for a multi-input system \( \dot{x} = Ax + Bu \), where \( u \in \mathbb{R}^m \). We pursue with computations in \( \mathbb{R}^n \) and inherit the notation form the last section. First, we present a basic fact, which generalizes the invariance statement referring to pole placement in Section 2.3 to the eigenstructure assignment problem.

**Fact 1** Consider the open-loop system (3). Let \( U = \text{Range}(U), \text{dim}(U) = r \leq m, \bar{X}_r \in \mathbb{R}^{r \times r}, \) and \( U^T B \) be full row rank. Introduce, further, a feedback control law \( u = K^T x \) and consider the closed loop system \( \bar{A} = A + BK^T \). Then:

(i) \( U \) represents a left eigenstructure of \( \bar{A} \) with the restriction \( \bar{X}_r \) thereon, i.e. \( U^T A = \bar{X}_r U^T \), if

\[
K^T = (U^T B)^+ (\bar{X}_r U^T - U^T A). \tag{39}
\]

(ii) If \( \mathcal{V} = \text{Range}(V) \) is \( A \)-invariant, i.e. \( AV = VY_{n-r} \) for some \( Y_{n-r} \in \mathbb{R}^{(n-r) \times (n-r)} \), then it is also \( \bar{A} \)-invariant under the feedback control (39) if \( U = \mathcal{V}^\perp \). Then:

\[
\bar{A} := A + BK^T = \begin{bmatrix} \bar{X}_r & 0 \\ Y_{n-r} \end{bmatrix} : U \to U, \quad \mathcal{V} \to \mathcal{V}. \tag{40}
\]

**Proof:** (i) Eq. (39) follows directly from \( U^T (A + BK^T) = \bar{X}_r U^T \) and the assumption that \( U^T B \) is full row rank, which guarantees the existence of \((U^T B)^+\).

(ii) With \( U \perp \mathcal{V} \), we have \( K^T \mathcal{V} = 0 \) and \( \bar{A}V = (A + BK^T)V = AV = VY_{n-r}, \) indicating that \( \mathcal{V} \) is \( \bar{A} \)-invariant, too. \( \square \)

In words, with the feedback gain (39) we are able to assign to the closed loop \( \bar{A} \) an arbitrary left invariant subspace \( U = \text{Range}(U) \) and any \( r \)-eigenvalues along the block-diagonal of \( \bar{X}_r \) in the sense of the real Schur decomposition. Furthermore, in general, imposing the left invariant subspace \( U \) may cancel the right \( A \)-invariance of a subspace \( \mathcal{V} \), unless \( U \) is selected to be complementary perpendicular to \( \mathcal{V} \), i.e. \( U \oplus \mathcal{V} = \mathbb{R}^n \) and \( U \perp \mathcal{V} \). As a consequence, and with the fact in hand that \( U^T U = I_r \), a sequential algorithm similar to that in the previous Section 2.4 can be adopted to carry out possible numerical issues in computing the inverse term \((U^T B)^+\) in Eq. (39).

In the following, we distinguish two cases.

3.1. Strictly-determined: \( r = m \)

In this case, \( U^T B \) must be non-singular. The closed loop system matrix \( \bar{A} = A + BK^T \) then reads:

\[
\bar{A} = (I - B(U^T B)^{-1} U^T) A + B(U^T B)^{-1} \bar{X}_m U^T. \tag{41}
\]

In analogy to the single input systems, cf. Eq. (11), the closed loop characteristic polynomial reads:

\[
det(\lambda I - \bar{A}) = \det(\lambda I - \bar{X}_m) \times \det(B^T \text{adj}(\lambda I - A^T U)) \times \det(U^T B)^{-1}. \tag{42}
\]
The remaining \( n - m \) eigenvalues \( \{\lambda_i\}_{i=m+1}^n \) are determined implicitly by the selection of \( U \), and are not directly assignable. In other words, \( U \) is a design parameter. A related design procedure will be proposed in Theorem 2.

3.2. Under-determined: \( m = r + 1 \)

The degrees of freedom at the input exceed now by one the dimension \( r \) of the invariant subspace \( \mathcal{U} \). We consider this as the general case, as \( r + 1 < m \) can always be recast in this seemingly special form by filling \( U \) with \( (m - r - 1) \) arbitrary columns that do not violate the full row rank condition of \( U^T B \). Hereby, we next combine the solution form for \( r = m \) and the algorithm for the partial pole placement from Section II, to accomplish a complete pole placement. That is, we sacrifice the dimension of the assigned subspace by one for the corresponding degree of freedom and apply it for the pole assignment task.

Therefore, let us explicitly extract an input channel \( b \) from \( B \), which will be determined in the sequel, yielding:

\[
\dot{x} = Ax + \tilde{B}u + bv.
\]

With \( v = k^T x \), we gain the system \( \dot{x} = \tilde{A}x + \tilde{B}u \), where \( \tilde{A} = A + bk^T \), wherein we now apply (39) for the partial eigenstructure assignment:

\[
K^T = (U^T \tilde{B})^{-1}\left(\bar{X}_{m-1}U^T - U^T \tilde{A}\right),
\]

yielding:

\[
\tilde{A} = (I - \tilde{B}(U^T \tilde{B})^{-1}U^T) \tilde{A} + \tilde{B}(U^T \tilde{B})^{-1}\bar{X}_{m-1}U^T.
\]

From (40), we further have:

\[
\begin{pmatrix}
U^T \\
V^T
\end{pmatrix} \tilde{A}(U, V) = \begin{pmatrix}
\bar{X}_{m-1} \\
0
\end{pmatrix},
\]

indicating that the remainder \( n - m + 1 \) eigenvalues \( \lambda(\tilde{A}) \setminus \lambda(\bar{X}_{m-1}) \) are shared by the matrix \( V^T Q \bar{A} V \), which can be rewritten in the “closed loop” form:

\[
V^T Q \bar{A} V = \hat{A} + \hat{b}k^T,
\]

with

\[
\hat{A} = V^T \bar{A} V, \quad \hat{b} = V^T Q b, \quad \hat{k}^T = k^T V,
\]

and

\[
Q := I - \bar{B}(U^T \tilde{B})^{-1}U^T.
\]

The matrix \( Q \) represents the oblique projector onto \( V \) [recall that \( U \oplus V = \mathbb{R}^n \) and \( U \perp V \)] with \( m - 1 \) eigenvalues equal to 0, and \( n - m + 1 \) eigenvalues equal to 1. Using the identity \( \hat{A}^k \hat{b} = V^T Q (A Q)^k b \) for any \( k \in \mathbb{N}_0 \), we have:

\[
C[\hat{A}, \hat{b}] = V^T Q \cdot C[A Q, b] \begin{pmatrix}
I_{n-m+1} \\
0
\end{pmatrix}. \tag{49}
\]

Furthermore, since \( V^T Q V = V^T V = I_{m-1} \) we have:

\[
\text{Rank} \ V^T Q = n - m + 1, \quad \text{which, in light of (49), leads to the following statement.}
\]

Proposition 1 The subsystem \( (\hat{A}, \hat{b}) \) is controllable whenever \( (A Q, b) \) is controllable, i.e.

\[
\text{Rank} \ C[A Q, b] = n. \tag{50}
\]

Note that this sufficiency statement is independent of \( V \). The opposite of the proposition’s statement must not necessarily hold: \( (A, b) \) may be controllable, even if \( (A Q, b) \) is not. It is also important to emphasize that for a given \( U \) which satisfies the invertibility condition of \( U^T \tilde{B} \) in (44), Eq. (50) represents a further design constraint which is to be employed for the proper discrimination of the single-input channel \( b \) out of \( B \). This is demonstrated in the example in Section 3.4.

Next, we design \( \hat{k} \) such that some \( n - m + 1 \) eigenvalues \( \{\lambda_i\}_{i=m}^n \) are assigned to \( \hat{A} + \hat{b}k^T \) by making use of the pole placement formula from Section II:

\[
\hat{k}^T = \gamma_{m-p}^T \hat{C} \hat{C}^{-1} q_p(\hat{A}), \quad p \in m_0 \tag{51}
\]

where \( \hat{C} = C[\hat{A}, \hat{b}] \) and \( \hat{C}_c = C[\hat{A}_c, \hat{b}_c] \). Finally, the feedback gain \( k \) with minimal \( L_2 \)-norm in (47) is computed by using the pseudo-inverse of \( V^+ = V^T \):

\[
k^T = \hat{k}^T V^T. \tag{52}
\]

Notice that this equation is in accordance with the parametrization (30) and the related geometrical interpretation.

To show that \( k \) is well-defined, i.e. it is independent of \( V \), substitute (51) and (49) herein, and use the identities \( VV^T = I - UU^T \) and \( V^T Q V = I \) to get:

\[
k^T = \gamma_{m-p}^T \hat{C} \left[ V^T Q \cdot C[A Q, b] \begin{pmatrix}
I_{n-m+1} \\
0
\end{pmatrix} \right]^{-1} \times \n
\times V^T Q q_p(A)(I - UU^T). \tag{53}
\]

Due to a rank deficiency of \( m - 1 \), the projection matrix \( Q \) can be factorized as

\[
Q = QR.
\]
where \( Q \in \mathbb{R}^{n \times (n-m+1)} \) is composed of \( n-m+1 \) orthonormal columns and \( R \in \mathbb{R}^{(n-m+1) \times n} \) is full rank. Then, after substituting \( V^TQ = V^TQR \) into the last equation, we get:

\[
k^T = s_m^T \tilde{c}_c \left[ RC[AQ, b] \begin{pmatrix} I_{n-m+1} & 0 \end{pmatrix} \right]^{-1} \times R q_p(A)(I - UU^T), \tag{54}\]

which is independent of the special basis set by \( V \).

### 3.3. Algorithm

In summary, our algorithm for assigning a left invariant subspace \( U = \operatorname{Range}(U) \subseteq \mathbb{R}^{n-1} \) corresponding to the block-triangular matrix \( \Sigma_{m-1} = \mathbb{R}^{(n-1) \times (m-1)} \) and the complete \( n \) eigenvalues reads as follows:

1. Extract the channels \( \tilde{b} \) and \((\tilde{B}, b)\) in accordance with (50);
2. Obtain \((\tilde{A}, \tilde{b})\) from (47) and compute \( k^T \) directly from (54), or, alternatively use (52) following the computation of \( k^T \) via (51); and
3. Compute \( K^T \) from (44) and reorder the inputs \( \tilde{u} = K^Tx \) and \( v = k^Tx \) in (43).

### 3.4. Numerical example

For illustration purposes, consider the multi-input system with

\[
A = \begin{pmatrix} 3 & -3 & -7 \\ 0 & -4 & 0 \\ 1 & 3 & -5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}. \tag{55}\]

The eigenvalues of \( A \) are \( \mu_1 = 2, \mu_2 = -4, \mu_3 = -4 \). Suppose that we want to assign the left eigenvector \( U = [-1, 1, 1]^T \) and its eigenvalue at \( \lambda_1 = 2 \), as well as \( \lambda_2 = -5 \) and \( \lambda_3 = -6 \). Following (50), we extract as \( b \) the second column in \( B \), i.e. \( b = [1, 0, 2]^T \) and \( \tilde{B} = [1, 1, 2]^T \) since \( \operatorname{Rank}[AQ, b] = 3 \) holds true. This implies complete controllability of the reduced order pair \((\tilde{A}, \tilde{b})\) in (47) and enables the complete eigenvalue assignment. Furthermore, we have:

\[
U^T = 0.577 \times [-1, 1, 1], \quad V^T = \begin{pmatrix} 0.408 & 0.817 & -0.408 \\ 0.707 & 0 & 0.707 \end{pmatrix},
\]

\[
\tilde{A} = \begin{pmatrix} 2 & -9 \\ 4 & -10 \end{pmatrix} \quad \text{and} \quad \tilde{b} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.
\]

Now use (51) with \( q_p(A) = \lambda_2 I - A \), to compute the state feedback \( \hat{k} = [6, 124, 707]^T \) for assigning \( \lambda_2 = -5 \) and \( \lambda_3 = -6 \) to the reduced-order closed loop system, and (52) to compute the state feedback gain \( k = [3, 5, -2]^T \). Finally, with \( U^T \tilde{B} \neq 0 \), we use (44) to compute \( K \) with \( A = A + b\hat{k}^T \) and \( \lambda = -2 \), this yielding \( K = [0.5, -4.5, -1]^T \) and the closed loop system matrix:

\[
\hat{A} = \begin{pmatrix} 6.5 & -2.5 & -10 \\ 0.5 & -8.5 & -1 \\ 8 & 4 & -11 \end{pmatrix}.
\]

### IV. Switched systems

Consider a finite set \( A := \{A_1, \ldots, A_N\} \) of real Hurwitz matrices (i.e. all eigenvalues are in \( \mathbb{C}_- \)) in \( \mathbb{R}^{n \times n} \). A switched linear system is defined as

\[
\Sigma_\sigma : \dot{x} = A_{\sigma(t)}x(t), \quad x(0) = x_0 \in \mathbb{R}^n \tag{56}\]

where \( \sigma : \mathbb{R}_0^+ \to \mathbb{N} \) is a piecewise right-continuous function, referred to as the switching signal between the constituent systems \( A_j \). Note that \( \sigma \) is an arbitrary switching signal, assuming a finite number of switchings within a fixed time interval. Uniform exponential stability of \( \Sigma \) implies existence of the numbers \( M \geq 1 \) and \( \beta > 0 \), such that \( \|\psi(t, x_0)\| \leq M e^{-\beta t} \|x(0)\| \) for any \( x(0) \in \mathbb{R}^n \), e.g. [6].

A quadratic function \( \phi(x) = x^T L x \), with \( L \) being a real symmetric matrix, is a common quadratic Lyapunov function (CQLF) of \( \Sigma \) if

\[
L > 0, \quad A_j^T L + L A_j < 0, \quad j \in \mathbb{N}. \tag{57}\]

If the state space is irrelevant, then \( L \) (rather than \( \phi(x) = x^T L x \)) is referred to as CQLF. While it is a standard fact that an LTI system \( \dot{x} = Ax \) has a QLF if and only if the matrix \( A \) is Hurwitz, for a switched system \( \Sigma_\sigma \) with Hurwitz constituents \( A_j \), the existence of a CQLF is only a sufficient condition for the exponential stability under arbitrary switching. Finally, it is a fact that the existence of a CQLF is invariant w.r.t. similarity transformations applied to constituents \( A_j \).

#### 4.1. Schur complement and perturbation

We remark that in our work hereafter we work in \( \mathbb{C}^n \). We begin with a simple useful fact.

**Fact 2** Let \( E \) and \( G \) be Hermitians. Then:

\[
M = \begin{pmatrix} -E & \varepsilon F \\ \varepsilon F^H & -\varepsilon G \end{pmatrix} < 0 \tag{58}\]
if and only if $E > 0$, $G > 0$ and

\[ 0 < \varepsilon^{1/2} < \sigma^{-1}(E^{-1/2}FG^{-1/2}) = \|E^{-1/2}FG^{-1/2}\|^{-1}. \tag{59} \]

**Proof:** Referring to Section I, $M_1 < 0$ if and only if $E > 0$ and $M/E < 0$, where $M/E$ represents the Schur complement of $E$ in $M$, implying

\[ -G + \varepsilon F^H E^{-1} F < 0, \tag{60} \]

or $G > \varepsilon F^H E^{-1} F > 0$ for any $\varepsilon > 0$. Herefor we have: $I > \varepsilon (G^{-1/2}F^H E^{-1/2})(E^{-1/2}FG^{-1/2})$, which directly implies (59). On the other hand, if $G > 0$, then for a sufficiently small $\varepsilon > 0$ [9]:

\[ \text{In}(-G + \varepsilon F^H E^{-1} F) = \text{In}(-G), \tag{61} \]

that is, $M/E = (0, n, 0)$, or $M/E < 0$, which, jointly with $E > 0$, implies $M < 0$. \[ \square \]

In other words, for each $\varepsilon > 0$ smaller than an upper bound, $M_1 < 0$ is guaranteed by $E > 0, G > 0$. That is, if $\varepsilon$ is seen as a structural perturbation, (59) provides a robustness measure and for a sufficiently large $\varepsilon$, the condition $M < 0$ gets violated.

Next, consider the situation with an off-diagonal additive perturbation $\Delta$ of $M$ in (58):

\[ \tilde{M} = \begin{pmatrix} -E & \varepsilon F + \Delta \\ \varepsilon F^H + \Delta^H & -\varepsilon G \end{pmatrix}. \tag{62} \]

Following Fact 2, $\tilde{M} < 0$ if and only if $E > 0, G > 0$ and

\[ \varepsilon^{-1/2}\|E^{-1/2}(\varepsilon F + \Delta)G^{-1/2}\| < 1. \tag{63} \]

It is hereof clear that in this case, the lower and upper bound on $\varepsilon$ becomes tighter as $\|\Delta\|$ increases, and for a sufficiently large perturbation $\|\Delta\|$, the perturbed $\tilde{M}$ will lose its negative definiteness, as no solution for $\varepsilon$ exists in (63). Indeed, in (63), increasing $\varepsilon$, i.e. reducing the left factor $\varepsilon^{-1/2}$ to compensate for the disturbing effects of $\|\Delta\|$ may not fulfill the required condition due to the simultaneously increasing term $\varepsilon F$. If we let $\Delta = \varepsilon \tilde{\Delta}$ with $\|\tilde{\Delta}\| = 1$, then condition (63) can be simplified at the price of a conservativeness:

\[ \varepsilon^{1/2} \leq \left( \|E^{-1/2}FG^{-1/2}\| + \|E^{-1/2}\|\|G^{-1/2}\| \right)^{-1}. \tag{64} \]

With reference to condition (59), the latter confirms that the $\varepsilon$-bounds corresponding to the perturbed model (62) are reduced. In other words, if $\|\Delta\|^{1/2}$ is greater or equal to the right-hand side of the latter equation, then $\tilde{M} \geq 0$ in (62) for all $\varepsilon$.

### 4.2. Nominal switched system

The following proposition follows directly as an application of Fact 2 to exponential stability of nominal switched linear systems [2]. We remark that a subspace $\mathcal{V}$ is $\mathbf{A}$-invariant, if it is $A_j$-invariant for all $j \in \mathbb{N}$.

**Proposition 2** Let $\mathcal{V}$ be $\mathbf{A}$—invariant, i.e. let all the constituents $A_j$ of the switched linear system (56) assume a common geometrical representation:

\[ A_j = \begin{bmatrix} X_j & 0 \\ C_j & Y_j \end{bmatrix}, \quad j \in \mathbb{N}, \quad \mathcal{U} \ni \mathcal{V} \ni \mathcal{U}, \tag{65} \]

where $X_j$ and $Y_j$ represent the restrictions of $A_j$ on $\mathcal{U}$ and $\mathcal{V}$, $X_j = A_j|_{\mathcal{U}}$ and $Y_j = A_j|_{\mathcal{V}}$. Let further $P > 0$ and $Q > 0$ be CQLFs for the corresponding switched subsystems corresponding to the constituents $X_j$ and $Y_j$, respectively. Then, there always exists an $\varepsilon > 0$, such that the function

\[ L = \begin{bmatrix} P & 0 \\ 0 & \varepsilon Q \end{bmatrix} \tag{66} \]

represents a net CQLF for the underlying switched system (56).

**Proof:** Obviously, $L = L^H > 0$, and in light of the lemma assumptions, we have:

\[ -E_j := X_j^H P + PX_j < 0, \tag{67} \]

and

\[ -G_j := Y_j^H Q + QY_j < 0. \tag{68} \]

With reference to Fact 2, there exists an upper-bounded $\varepsilon > 0$, such that

\[ A_j^H L + L A_j = \begin{bmatrix} -E_j & \varepsilon C_j Q \\ \varepsilon Q C_j^H & -G_j \end{bmatrix} < 0, \quad j \in \mathbb{N}. \tag{69} \]

The upper bound of $\varepsilon$ results now from Eq. (59):

\[ 0 < \varepsilon^{1/2} < \min_{j \in \mathbb{N}} \|E_j^{-1/2}C_j Q G_j^{-1/2}\|^{-1}. \tag{70} \]

It is important to emphasize that thereby, in addition to the existence of the CQLF for the switched subsystems $X_j$ and $Y_j$, the critical requirement is the $\mathbf{A}$—invariance, i.e. the same geometrical structure of the constituents, sharing a common right invariant subspace $\mathcal{V}$. In this sense, twofold conditions are here required.

Proposition 2 in conjunction with Fact 1 provide now the basis for a stabilizing control algorithm for a
particular class of switched systems, which we state in form of a theorem. We remark that the local controller gains $K_j$ in the following theorem refer to the feedback law $u = P^T_i x$, where $\sigma$ indicates switching between the different gains $K_j$, as triggered by $\sigma(t)$ in accordance with the assumptions and definitions in (56). Clearly, a gain $K_j$ will correspond to the open-loop constituent pair $[A_j, B_j]$ in the model (71).

**Theorem 2** Consider the definitions as in (56) and an open-loop switching linear system given by

$$\Sigma_{\sigma,\alpha} : \dot{x} = A_{\sigma(t)} x(t) + B_{\sigma(t)} u, \quad (71)$$

with $u \in \mathbb{R}^m$, $B_j \in \mathbb{R}^{n \times m}$. Let all the constituents share a common invariant subspace $V := \text{Range}(V)$, i.e., $A_j V = V B_j$ and, additionally, let all $Y_j$ share a CQLF. Then, $\Sigma_{\sigma,\alpha}$ is exponentially stabilizable by state-feedback $u = K_j^T x$ if and only if for all $i, j \in \mathbb{N}$

$$\text{Rank} \left( B_i^T Q_j B_j \right) = m, \quad (72)$$

where $Q_j := I - V (VH)^{-1} VH$. Moreover, with $U \perp V, U \oplus V = \mathbb{C}^n$ and $U = \text{Rank}(U)$, we have:

$$K_j = (UH B_j)^+ (XU^H - UH A_j), \quad (73)$$

where $X$ is some asymptotically stable matrix and $U \in \mathbb{C}^{n \times m}$ is given by

$$U = Q_j B_i \text{ for a fixed } i \in \mathbb{N}. \quad (74)$$

**Proof:** It is easy to see that the feedback controller (73) enforces the conditions of Proposition 2 if a basis $U$ of $U = \text{Range}(U)$ exists satisfying the two conditions:

$$U^H V = 0, \quad \text{and } \text{Rank} \left( U^H B_j \right) = m, \quad j \in \mathbb{N}. \quad (75)$$

Thus, it suffices to discuss the existence of such a $U$.

$(\Rightarrow)$ Fix an $i \in \mathbb{N}$. Then, $U^H V = 0$ follows immediately if $U = Q_j B_i$. Setting $i = j$ in (72) implies $B_j^T Q_j B_j = UH B_j$, leading to $\text{Rank} \left( UH B_j \right) = m$ for all $j \in \mathbb{N}$.

$(\Leftarrow)$ Let $j = i$ in (75). Since $UH Q_i B_i = UH B_i$, and since we assume that $\text{Rank} \left( UH B_i \right) = m$, the rank of $Q_i B_i$ cannot be lower than $m$, implying that $\text{Rank} \left( Q_i B_i \right) = m$ must hold true. Then, given that $UH V = 0$ and $B_i^T Q_i V = 0$, it follows that $U$ and $Q_i B_i$ both span $\text{Kernel}(VH)$. As they are also both full rank, we must have $U = Q_i B_i M$ for some non-singular matrix $M \in \mathbb{C}^{m \times m}$. As a consequence, $m = \text{Rank} \left( UH B_j \right) = \text{Rank} \left( MH B_j^T Q_j B_j \right)$ implies Eq. (72): $\text{Rank} \left( B_j^T Q_j B_j \right) = m$ for all $j \in \mathbb{N}$. Eq. (74) follows if we let $M = I_m$. □

Few comments are now in order.

(i) From the proof of Proposition 2 it is obvious that $U$ and $V$ need not necessarily be perpendicular. Yet, in accordance with Fact 1, in the above theorem we pose perpendicularity for to preserve the common right invariance of $V$ after left eigenstructure assignment.

(ii) The subsystems $A_j, j \in \mathbb{N}$, in the above theorem are assumed to respect the representations:

$$A_j = \begin{bmatrix} X_j & 0 \\ C_j & Y_j \end{bmatrix} : U_j \oplus V \rightarrow \oplus V, \quad (76)$$

i.e. with different subspaces $U_j$ corresponding to the different constituents $A_j$. In this picture, $U_j$ and $V$ are complementary, but not necessarily mutually orthogonal. In fact, the control task solved by the above theorem consists precisely in assigning a left common invariant ysubspace $U \perp V$ to all the constituents in the closed loop, as well as the assignment of a corresponding (perhaps, common) left partial eigenstructure $X := X_j$ along $U = \text{Range}(U)$, as claimed by (67) for some arbitrary $P = P^H > 0$ (perhaps, $P = I$). In this way, both conditions required by Proposition 2 are guaranteed.

(iii) Condition (72) represents a restriction in applying the proposed design procedure of this section. Although it does specify a mutual relationship between the geometry (of the system) and the control (input matrices), it is not related to the controllability of the modes of the original system (56). Yet, it is stronger than the controllability of the pair $(X_j, B_{1j})$, arising in the similar forms of (71):

$$\left( \begin{array}{c} \dot{z} \\ \zeta \end{array} \right) = \begin{bmatrix} X_j & 0 \\ * & Y_j \end{bmatrix} \left( \begin{array}{c} z \\ \zeta \end{array} \right) + \begin{bmatrix} B_{1j} \\ B_{2j} \end{bmatrix} u, \quad (77)$$

as $B_{1j} := UH B_j$ is requested to be invertible.

(iv) If all the modes of the switching system share the same system input matrix $B$, then the set of conditions (72) simplifies to one condition only:

$$\text{Rank} \left( Q_j B \right) = m. \quad (78)$$

(v) Finally, in accordance with (74), the closed form state-feedback control reads:

$$K_j^T = (B_j^T Q_j B_j)^{-1} (X B_j^T Q_j - B_j^T Q_j A_j). \quad (79)$$

4.3. Algorithm

Summarizing, our algorithm follows the lines:
Given an open-loop switched linear system in the form (56) with $m$ degrees of freedom at the input, identify a common invariant subspace $\mathcal{V}$ spanned by $V \in \mathbb{C}^{n \times (n-m)}$ in accordance with (76) for all $j \in \mathcal{N}$ (for instance, the algorithm from [20] can be used therefor);

(2) Check whether the subsystems $Y_j$ share a CQLF;

(3) Verify the conditions (72) for some fixed $i \in \mathcal{N}$ and all $j \in \mathcal{N}$; and

(4) If confirmative, then set $U = Q \bar{Y}_j$ for some $i \in \mathcal{N}$ and compute the feedback controllers using (78).

### 4.4. Numerical example (cont.)

Consider the problem data of the numerical example in Section 3.4, adopted from [20]:

$$A_1 = \begin{pmatrix} 3 & -3 & -7 \\ 0 & -4 & 0 \\ 1 & 3 & -5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 3 & -5 \\ 2 & -6 & -2 \\ 7 & 1 & -11 \end{pmatrix}$$

and

$$B_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$  

The modes $A_1$ and $A_2$ share a common invariant subspace spanned $\mathcal{V}$ by $\bar{V}$ with:

$$V = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} -4 & 4 \\ 0 & -4 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} -4 & 8 \\ 0 & -4 \end{pmatrix}.$$  

As $Y_1$ and $Y_2$ are Hurwitz, we may take $Q = I_2$ in (66). In accordance with the algorithm in Section 4.3, we select $U^T = [-1, 1, 1]$. As $B_j^T Q_j B_j$ are full rank for all $i, j = 1, 2$, we can impose $U$ to $\bar{A}_1$ and $\bar{A}_2$, e.g. using $X = -2$. The resulting state feedbacks $K_1 = [2, -2, -2]^T$ and $K_2 = [-6, 6, 6]^T$, obtained from (78), provide the closed loop matrices:

$$\bar{A}_1 = \begin{pmatrix} 5 & -5 & -9 \\ 2 & -6 & -2 \\ 5 & -1 & -9 \end{pmatrix}, \quad \bar{A}_2 = \begin{pmatrix} -5 & 9 & 1 \\ 2 & -6 & -2 \\ -5 & 13 & 1 \end{pmatrix},$$  

which share the same invariant subspace $V$ and a common left invariant eigenvector $U$. The corresponding switched system in (56) is then exponentially stable.

### V. Robustness

The requirement for a common right invariant subspace in Proposition 2 is apparently restrictive. In this section, we drop this condition and investigate the stability of the switched system (56), with its constituents sharing a common invariant subspace approximately. More precisely, we introduce perturbed models $\bar{A}_j$ of $A_j$, involving right invariant subspaces $\bar{V}$ that are sufficiently “close” to each other. The “distance” between subspaces is measured by means of the gaps $\theta_{\text{max}}$, as indicated in Figure 1, cf. Eq. (6).

![Perturbation of the common invariant subspace $\mathcal{V}$.

#### 5.1. Perturbation of invariant subspaces

A natural question to consider here is for what maximal perturbation $\theta_{\text{max}}$ of $\mathcal{V}$ to $\bar{Y}_j := \bar{A}_j Y_j$, the CQLF (66) remains preserved. $\bar{A}_j Y_j$ will, of course, lie outside $\mathcal{V}$, indicating that we need to modify the condition $\bar{A}_j V = Y_j V$ in (76).

**Proposition 1** Introduce an arbitrary matrix $\Delta$ of proper dimensions and consider the perturbation model:

$$\bar{A}_j V = (V + U \Delta_j) Y_j. \quad (79)$$

Then: $\tan \theta_{\text{max}, j}(\mathcal{V}, \bar{Y}_j) = \|\Delta_j\|$.  

**Proof:** First, after introducing $\bar{V} \equiv \bar{A} V$ (for simplicity, we drop $j$-indexing), note that the oblique projection $\mathcal{P}$ onto $\bar{V} = \text{Range}(\bar{V})$ along $U$, given by:

$$\mathcal{P} = \bar{V} (V^H \bar{V})^{-1} V^H \quad \text{use (79)}$$

satisfies

$$\mathcal{P}^H \mathcal{P} = V Y^{-H} \bar{V}^H \bar{V} Y^{-1} V^H = V (I + \Delta^H \Delta) V^H$$

and

$$\|\mathcal{P}\|^2 = 1 + \|\Delta\|^2 = 1 + \sigma_{\text{max}}^2 (\Delta). \quad (80)$$

On the other hand, with $U = \text{Span}(U)$ being complementary orthogonal to $\mathcal{V}$, and considering (8), one
For $\Delta$ Compared to the non-perturbed model (28), a coupling assume that the subsystems preserved. Additionally, note that we continue to is a hope yet that the CQLF in the form (66) remains implied $\tan \theta_{\max}(\mathcal{V}, \tilde{V}) = \sigma_{\max}(\Delta) = ||\Delta||$. □

The perturbation model (79) leads to the following perturbed switched system:

$$\tilde{A}_j = \begin{bmatrix} X_j & \Delta_j Y_j \\ C^H_j & Y_j \end{bmatrix} : \mathcal{U} \rightarrow \mathcal{U}, \ j \in \mathcal{N}. \quad (82)$$

Compared to the non-perturbed model (28), a coupling term $\Delta_j Y_j$ from $\mathcal{V}$ to $\mathcal{U}$ arises now. Intuitively, there is a hope yet that the CQLF in the form (66) remains preserved. Additionally, note that we continue to assume that the subsystems $X_j$ in the perturbed models still share a CQLF, which is indeed a non-critical assumption. In fact, in the next section we provide a feedback control regularization formalism which guarantees these conditions. That said, the perturbation model (79) refers solely to perturbation of the geometry of the system.

**Lemma 1** Consider the perturbed system (82), and let $P$ and $Q$ represent CQLFs for the subsystems $X_j$ and $Y_j$, $j \in \mathcal{N}$, respectively, cf. Proposition 2.

(i) $L = \text{diag}(P, \varepsilon Q)$ is preserved in (82) if and only if for all $j \in \mathcal{N}$

$$\varepsilon^{-1/2} ||E_j^{-1/2}(\varepsilon C_j Q + P \Delta_j Y_j)G_j^{-1/2}|| < 1. \quad (83)$$

(ii) For $\Delta_j = \varepsilon \tilde{\Delta}_j$ with $||\tilde{\Delta}_j|| = 1$ the CQLF $L = \text{diag}(P, \varepsilon Q)$ is preserved by (82) if and only if

$$\tan \theta_{\max}(\mathcal{V}, \tilde{V}_j) < \frac{1}{\sigma} \text{ for all } j \in \mathcal{N} \quad (84)$$

where

$$\sigma = \max_{j \in \mathcal{N}} \max_{||\Delta_j|| \leq 1} ||E_j^{-1/2}(C_j Q + P \tilde{\Delta}_j Y_j)G_j^{-1/2}||. \quad (85)$$

**Proof:** With reference to the proof of Proposition 2, we have:

$$\tilde{A}_j^H L + L \tilde{A}_j = \begin{bmatrix} -E_j & \varepsilon C_j Q + P \Delta_j Y_j \\ \varepsilon Q C^H_j + Y_j^H \Delta_j^H P & -\varepsilon G_j \end{bmatrix}. \quad (86)$$

(i) Then (83) is a restatement of (63).

(ii) In light of Proposition 1 the parameter $\varepsilon$ in $L = \text{diag}(P, \varepsilon Q)$ can be interpreted as the perturbation of the invariant subspace $\mathcal{V}$. The statement of lemma follows now directly from Proposition 1 and Fact 2, given that $\tan \theta_{\max}(\mathcal{V}, W_j) = \varepsilon ||\Delta_j|| \leq \varepsilon$. □

Note that the statement (ii) is based on coupling of the Lyapunov function $L = \text{diag}(P, \varepsilon Q)$ to the system uncertainty in that the parameter $\varepsilon$ is associated with the value of the uncertainty norm.

In a similar manner, we can derive a corresponding robustness lemma with respect to the perturbation of the left invariant subspace $\mathcal{U}$. Analogously to Proposition 1, one can show that a perturbation defined by $U^H \tilde{A}_j = X_j(U^H + V^H \Delta_j^H)$, i.e., by

$$\tilde{A}_j = \begin{bmatrix} X_j & \Delta_j^H \\ C^H_j & Y_j \end{bmatrix} : \mathcal{U} \rightarrow \mathcal{U}, \ j \in \mathcal{N}, \quad (86)$$

corresponds to the perturbation $\tilde{U}_j$ of the left invariant space $\mathcal{U}$ by the gap $\tan \theta_{\max}(U, \tilde{U}_j) = ||\Delta_j||$. Then, $L = \text{diag}(P, \varepsilon Q)$ is a CQLF for the perturbed switched system (86) if and only if for all $j \in \mathcal{N}$, cf. Eq. (83):

$$\varepsilon^{-1/2} ||E_j^{-1/2}(\varepsilon C_j Q + P X_j \Delta_j^H)G_j^{-1/2}|| < 1. \quad (87)$$

### 5.2. High-gain regularization

In the previous section, we have presumed unperturbed subsystems $X_j : \mathcal{U} \rightarrow \mathcal{U}$ sharing all a common QLF. With reference to Fact 1, in this section we apply the state-feedback assignment technique of Section III to guarantee this condition. Recall that the control action (39) will take place in the subspace $\mathcal{U}$, and it has no effect in the subdynamics on the subspace $\mathcal{V}$, a fact that we also used in the design of controllers in the nominal case in Section 4.2. In this section, we show that the latter holds true also in the perturbed case, i.e., for the systems of the form (82). To this end, consider an open-loop perturbed switching system

$$\tilde{\Sigma}_\sigma : \dot{x} = \tilde{A}_j x(t) + B_\sigma u(t), \quad (88)$$

with $\sigma \rightarrow j \in \mathcal{N}$, $\tilde{A}_j \in \mathbb{R}^{n \times n}$, $B_\sigma \in \mathbb{R}^{n \times m}$, and let all its constituents respect the geometry:

$$\tilde{A}_j = \begin{bmatrix} \tilde{X}_j & \Delta_j^H \\ \tilde{C}_j^H & Y_j \end{bmatrix} : \mathcal{U} \rightarrow \mathcal{U}, \ j \in \mathcal{N}, \quad (89)$$

where for the sake of generality we assume perturbed terms $\tilde{X}_j$ and $\tilde{C}_j$, cf. Eq. (82). Similarly, as in
Section 4.2 we apply local feedback controllers $u = K^T \sigma x$, with
\[ K^T = (U^H B_j)^+ (X_j U^H - U^H A_j). \quad (90) \]
A controller $K_j$ is aware of the nominal system only, hence the term $A_j$ must appear therein. Also, $K_j$ must be real, which is ensured if $U$ and $V$ in Proposition 2 correspond to conjugate multisets of eigenvalues of $A_j$. The corresponding closed loop:
\[ \tilde{A}_j = \hat{A}_j + B(U^H B)^+ (X_j U^H - U^H A_j), \quad (91) \]
assumes indeed the closed loop dynamics (82) with:
\[ \tilde{X}_j := U^H \tilde{A}_j U = X_j + U^H \tilde{\Delta}_j U, \quad (92) \]
which obviates from $X_j$ due to the perturbation of the nominal system $\tilde{\Delta}_j := \hat{A}_j - A_j$. Further:
\[ U^H \tilde{A}_j V = \Delta_j Y_j, \quad \text{and} \quad V^H \tilde{A}_j V = Y_j. \quad (93) \]
Notably, the feedback law (90) affects only the first block-column and has no effect on the second column of the open loop, cf. (94). Hence, if we manage to find a $P$ which is a CQLF for the subsystems $\tilde{X}_j : U \to U$, then we succeed in recovering the conditions in Lemma 1. But, we can always select $P = I$ if we assume that $X_j$ (as designed in Section III) possesses a real eigenvalue $\lambda_k < 0$ such that $|\lambda_k| > \max_{j \in \mathcal{N}} \|\tilde{\Delta}_j\|$.

Note that such a control action can be used to increase the norm of the term $E_j$ in Eq. (64) and hence increase the robustness margins with respect to the perturbation of the invariant subspace $\mathcal{V}$. With reference to (90), this represents clearly a high-gain control action. This is an interesting fact, since all $X_j$ are free parameters in our control approach. In other words, selection of a sufficiently large smallest value in magnitude in the set $\Lambda(X_j), j \in \mathcal{N}$, can compensate for the mismatches of the constituents’ right invariance subspaces $\tilde{\mathcal{V}}_j$.

5.4. Stable invariant subspaces

Lemma 1 reveals the impact of the perturbation of the geometry into the stability of a switching system. By making use of the concept of stable invariant subspaces, we can extend the class of perturbations. We call a common $A$-invariant subspace $\mathcal{V}$ stable if for any $\varepsilon > 0$, there exists $\delta > 0$ and $\tilde{V}$ such that $\|\tilde{A}_j - A_j\| < \delta$, $j \in \mathcal{N}$, implies $\theta(\mathcal{V}, \tilde{\mathcal{V}}) < \varepsilon$, [e.g., see 7], where $\theta(\cdot)$ refers to some subspace gap. In words, stable invariant subspaces are preserved despite a perturbation of $A$, and therefore there exists a transformation such that all perturbed constituents share the picture:
\[ \tilde{A}_j = \begin{bmatrix} X_j + \tilde{\Delta}_j & 0 \\ \tilde{C}_j H & Y_j + \tilde{\Delta}_j \end{bmatrix} : \tilde{U} \to \tilde{U}, \quad (94) \]

VI. Triangularization

In this section we collect generalizations of a few of the ideas developed in the previous two sections. We show that the adopted geometrical approach leads to new insight and more elegant proofs of some well-known stability results of triangularized switching systems and identify a new stable subclass. The basic concept we use here is that of complete invariant subspaces, which we introduced in Section I. We start with a slight technical extension of Fact 2.

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Fact 3 Let $E, G_i$ be Hermitians, and define $M_0 := -E$, $M_i$ as the $i \times i$ upper triangular of $M_m$:

$$M_m = \begin{pmatrix} -E & \varepsilon F_{11} & \ldots & \varepsilon F_{1m} \\ -\varepsilon F_{11}^H & -\varepsilon F_{11} G_1 & \ldots & \varepsilon F_{1m} G_1 \\ \ldots & \ldots & \ldots & \ldots \\ \varepsilon F_{m1} & \varepsilon F_{m1}^H & \ldots & -\varepsilon F_{m1} G_m \\ \varepsilon F_{m1} & \varepsilon F_{m1}^H & \ldots & -\varepsilon F_{m1} G_m \end{pmatrix}$$  \ (95)

and $F_{i}^H = [F_{i1}^H, \ldots, F_{im}^H]$, for $i \in m$. Then, $M_m < 0$ if and only if $E > 0, G_i > 0$, and

$$0 < \varepsilon_i^{1/2} < \|M_{i-1}^{-1/2} F_i G_i^{-1/2}\|^{-1}, \ i \in m.$$  \ (96)

Proof: Introduce the following partitioning of $M_m$:

$$M_m = \begin{pmatrix} M_{m-1}(\varepsilon_1, \ldots, \varepsilon_{m-1}) & \varepsilon_m F_{m} \\ \varepsilon_m F_{m}^H & -\varepsilon_m G_m \end{pmatrix}.$$  

Fact 2 implies that $M_m < 0$ if and only if $M_{m-1} < 0$, $G_m > 0$ and $\varepsilon_m > 0$ is sufficiently small and satisfies (96). The statement results after an iteration of the same reasoning on $M_{m-1}, M_{m-2}, \ldots$, all the way down to $M_1$. □

Notice that this fact and (96) provide an iterative procedure for the computation of the upper bounds for the $\varepsilon$-parameters in that $\varepsilon_i = \varepsilon_i(\varepsilon_1, \ldots, \varepsilon_{i-1}), \ i \in m$, with $\varepsilon_0 = 0$. The bounds of $\varepsilon$-parameters are computed in the order $\varepsilon_1, \varepsilon_2$ up to $\varepsilon_m$.

Proposition 2 Let the constituents of the switching system (56) assume the geometrical representation:

$$A_j = \begin{bmatrix} X_j & 0 & \ldots & 0 \\ C_{1j}^H & Y_{1j} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_{mj}^H & C_{mj}^H & \ldots & Y_{mj} \end{bmatrix} \oplus \oplus \oplus \oplus \oplus \oplus \oplus \oplus$$  \ (97)

and let the subsystems $X_j, Y_{ij}, j \in N$, share the CQLFs $P_i, Q_i, i \in m$, respectively. Then, there always exist numbers $\varepsilon_i > 0, i \in m$, such that

$$L = \text{diag}(P, \varepsilon_1 Q_1, \ldots, \varepsilon_m Q_m)$$  \ (98)

represents a CQLF of the switching system (97).

Proof: The proof follows similar lines as that of Proposition 2 and makes use of Fact 3. □

Again, the upper bounds for $\varepsilon_i, i \in m$, are iteratively computable and they follow from (70):

$$0 < \varepsilon_i^{1/2} < \min_{j \in N} \|M_{i,j}^{-1/2} C_{i,j} Q_i G_{i,j}^{-1/2}\|^{-1},$$  \ (99)

where

$$-G_{i,j} := Y_{i,j}^H Q_1 + Q_i Y_{i,j} < 0,$$

$$C_{i,j}^H = [C_{i,j}^H, \ldots, C_{i,j}^H],$$

$$M_{i,j} = \begin{pmatrix} M_{i-1,j}(\varepsilon_i, \ldots, \varepsilon_{i-1}) & \varepsilon_{i-1} C_{i-1,j}^H Q_{i-1} \varepsilon_{i-1} \varepsilon_{i-2} \ldots \varepsilon_{i-1} \end{pmatrix}.$$  \ (100)

Again, a recursive algorithm for computation of the bounds of $\varepsilon_i, i \in N$ is invoked, since $M_{i,j} = M_{i,j}(\varepsilon_i, \ldots, \varepsilon_{i-1})$, with $M_{0,j} := -E_j$. Proposition 2 is useful in that it leads to new insights and proofs of some well-known theorems about the existence of CQLFs for few special cases of switched systems.

Theorem 3 Let all $A_j$ of $\Sigma_\sigma$ share a complete chain of invariant subspaces $\emptyset = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = \mathbb{C}^n$, where $\dim M_i = i, i \in n$. Then, there exists a CQLF for the system $\Sigma_\sigma$ in the form:

$$L = \text{diag}(1, \varepsilon_1, \ldots, \varepsilon_{n-1}).$$  \ (101)

Proof: In accordance with the Schur-decomposition, the terms $Y_{ij}$ are now scalars equal to the eigenvalues $\lambda_{ij} \in \mathbb{C}$ of $A_j$. Hence, in this case, we can pick $P = 1$ and $Q_i = 1$ for every $i$ in (98). □

Corollary 1 For every following class of the switched system $\Sigma_\sigma$ (56) there exists a CQLF of the form (101):

(C1) $\{A_j\}_{j \in N}$ is a collection of commutative matrices, that is, $A_p A_q = A_q A_p$ for all $p, q \in N$;

(C2) The elements of $\{A_j\}_{j \in N}$ are simultaneously lower triangularizable; and

(C3) Every pair in $\{A_j\}_{j \in N}$ has a rank 1 commutator, that is, $\text{Rank} (A_p A_q - A_q A_p) \leq 1, p, q \in N$.

Proof: For the classes (C1) and (C2) there exists a common complete chain of invariant subspaces [7]. The class (C3) is slightly more subtle, as only pairwise complete chains exist. That is, for any pair $(A_p, A_q)$, there exists a CQLF, say $L_{pq}$, of the form (101), [again, see, 7]. But, we can then always construct a CQLF $L$ (101) using

$$\varepsilon_i \leq \min_{p,q \in N} L_{pq}(i + 1, i + 1), \ i \in \{1, \ldots, n - 1\}.$$  

Eq. (96) guarantees that such a diagonal matrix represents a CQLF for $\Sigma_\sigma$. □

Note that the classes (C1) and (C2) are well-known in the literature of switching systems. For instance, the class (C1) was firstly introduced in [15] and (C2) in [14]. Corollary 3 indicates that they are, in fact,
equivalent classes, which is clear as it is well known fact that a set of commuting matrices is simultaneously similar to a set of lower-triangular commuting matrices. On the other hand, (C1) clearly a subclass of (C3). We refer also to the result [8], where it is shown that a pair of matrices with rank one commutator is simultaneously triangulizable.

VII. Conclusions

We develop analytical underdetermined state-feedback design algorithms and analysis tools for diverse control problems including spectrum assignment, eigenstructure assignment and exponential stabilization of switched linear systems by utilizing computational methods based on the concept of invariant subspaces. Moreover we explore the usability of specific design and analysis tools in the respective growing problem tasks. While in the first two problem settings we focus on the design issues, in the latter case, we derive induction statements about the existence of a net common quadratic Lyapunov function for the class of switched systems with the constituents sharing specific geometric and structural properties. These involve basically a common right invariant subspace and a common quadratic Lyapunov function referring to the subsystems thereupon. Such stringent conditions are relaxed in a further step by invoking perturbations of the common geometry, this indicating the effects of the geometry into the system stability. A deeper understanding of the role of the stable invariant subspaces in the net stability of the switched linear systems is an appealing research aspect.

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