Lyapunov stability bounds in the controller parameter space

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Abstract—A novel Lyapunov stability based method for the parameter space design of linear time invariant control systems is proposed in this paper. The basic idea lies in utilizing the Lyapunov equation as the basic vehicle for mapping the stability bounds into the control parameter space. In comparison to the existing parameter space methods in the literature, the new technique provides two computationally critical advantages. First, it avoids the necessity for the frequency gridding which has been almost traditional and a source of an inherent computational complexity. Second, the proposed design method provides directly the bounds in the multi-dimensional control parameter space, thereby dropping the limitation of most related design techniques that take place on a parameter plane. The usage of the proposed method is demonstrated in the prominent case study of a PID control.

I. INTRODUCTION

The problem of computing all stabilizing controller parameters can be traced back to Vyshnegradsky [1] and has been investigated during last decades [2], [3], [4], [5]. Several methods are presented to compute the stable regions in the space of the controller coefficients. Among them, there are the D-decomposition method [5], [6], the parameter space method, the singular frequency method and the Hermite-Biehler approach. All these methods are based on the fact that a linear system gets unstable when at least one root of its characteristic equation crosses to the right half plane. In the parameter space method, stability critical controller coefficients are the ones where the roots of the characteristic polynomial cross the imaginary axis or change signs at infinity. They are defined as stability crossing boundaries. The accuracy of the computed boundaries depends on the step size of the frequency sweeping. Sometimes the frequency dependency is eliminated between the real and imaginary equations to extract the controller coefficients explicitly. Such an approach is followed, e.g. by the singular frequency method [3], [7], whereby the frequency sweeping step is dropped efficiently particularly in the case of PID control. In this method, the stability crossing boundaries are straight lines (singular lines) in the \( K_I-K_D \) plane. Each singular line corresponds to one singular frequency. The singular lines and singular frequencies are computed from two equations which are obtained by implementing mathematical operations on the real and imaginary parts of the characteristic equation.

The Hermite-Biehler method uses the same idea to derive two equations from the characteristic equation such that one of them is independent of \( K_P \) and the other is independent of \( K_I \) and \( K_D \) [4]. However, in this approach the phase specifications of the characteristic equation are employed to compute the stability crossing boundaries, using the fact that the roots of the real and imaginary parts of a stable polynomial interface. For the case of time delay systems, the stability crossing boundaries are computed in the space of controller coefficients using the singular frequency method [7], [9] and the Hermite-Biehler method [10]. A detailed comparison of the Hermite-Biehler method and the singular frequency method is presented in [8]. A disadvantage of the latter methods becomes critical in control scenarios with mesh control loops or higher dimensional controller parameter spaces [11].

In the present paper we propose a Lyapunov equation based approach which dispenses with the frequency dependent stability boundaries. Furthermore, the approach is directly applicable for the MIMO systems, as well, and it does not confine to a parameter plane, i.e. it does not necessitate a parameter space griding, as it has been the case in the case of computation of stable PID region, e.g., see [2]. Note that similar ideas covering the dependency of the eigenvalues of the system and corresponding Lyapunov matrices are discussed in [14], but their discussion is not linked to a specific design procedure. The remainder of the present article is organized as follows. In the next section the basic concept of the Lyapunov based stability margin calculation is introduced. In Section III, the novel Lyapunov based approach is demonstrated for a PID case study. The final section gives a summary of the main facts of this paper.

II. LYAPUNOV EQUATION BASED MAPPING TECHNIQUE

In this section, a descriptive introduction of the design technique is proposed. Consider a linear time-invariant (LTI) state-space model of a closed-loop system \( \dot{x} = A(k)x \) where \( x \in \mathbb{R}^n \), \( A \in \mathbb{R}^{n \times n} \) and \( k \in \mathbb{R}^p \) referring to the control parameters. For instance, in the case of PID control it holds \( k = (K_P, K_I, K_D)^T \). Notice that the matrix \( A \) is assumed in its controller canonical form for simplicity. Hence, the last row of \( A \) hosts explicitly the coefficients of the characteristic polynomial which is particularly convenient for the design of the controller parameters within the proposed state-space design framework. From a Lyapunov point of view the system \( \dot{x} = A(k)x \) is asymptotically stable if and only if the Lyapunov equation

\[
A^T(k)P + PA(k) = -Q, \tag{1}
\]
is feasible, i.e. if it holds true for some strictly positive definite matrices $P$ and $Q \in \mathbb{R}^{m \times m}$, e.g., see [12] with

$$P = \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nn} \end{pmatrix}. $$

On the other hand, it is a fundamental result in algebra that $P > 0$ if and only if the determinants of all its leading principal minors are strictly positive, e.g., see [13]. The leading principal minors $\mu_i$ of $P$ refers to the upper left $i \times i$ corner of $P$ for $i = 1, \ldots, n$

$$\mu_i = \begin{pmatrix} p_{11} & \cdots & p_{i1} \\ \vdots & \ddots & \vdots \\ p_{i1} & \cdots & p_{ii} \end{pmatrix}. \quad (2)$$

Note that (1) can be rewritten in the vector form as

$$(I \otimes A^T(k) + A^T(k) \otimes I) \text{vec}(P) = -\text{vec}(Q), \quad (3)$$

where $I$ is a $n \times n$ identity matrix, $\otimes$ is the Kronecker product and $\text{vec}(-)$ is the vector rearranging the matrix entries column-after-column. Hence, the entries of $P$ can be computed from

$$\text{vec}(P(k)) = M^{-1}(k) \text{vec}(-Q), \quad (4)$$

where

$$M(k) = I \otimes A^T(k) + A^T(k) \otimes I. \quad (5)$$

The denominators of the entries of the matrix $P$ are given by the determinant $|M(k)|$ which is parameterized by the controller parameters $k$. It is important to observe that this determinant is a polynomial expression in terms of the parameters of $k$, unless the open-loop transfer function, i.e. $K(s)G(s)$ is proper, where $K(s)$ refers to the controller and $G(s)$ to the plant. In the Latter case, the relative degree of $K(s)G(s)$ is zero which means that $|M(k)|$ is a rational function of $k$, i.e. in the form

$$|M(k)| = \frac{m_n(k)}{m_d(k)}, \quad (6)$$

where the numerator $m_n(\cdot)$ and denominator $m_d(\cdot)$ are polynomial expressions.

To get a better insight into the discussed matter, consider a feedback loop with a PID controller $K(s) = K_P + K_I/s + K_D s$ and a plant transfer function $G(s) = N(s)/D(s)$ where $N(s)$ and $D(s)$ are polynomials of degrees $m$ and $n$, respectively. The degrees of the numerator and the denominator of the open loop transfer function $G_0(s) = G(s)K(s)$ read then $m+2$ and $n+1$, i.e. the underlying transfer function is strictly proper for $n > m+1$ and proper for $n = m+1$. Hence, in the former case, it is expected $|M(k)|$ to be a pure polynomial expression whereas in the latter case it is a rational one. For a concrete example, let's consider $G(s) = 1/(s + 1)$. Then, the characteristic equation of the unity feedback loop with the PID controller is $(1 + K_D) s^2 + (1 + K_P) + K_I = 0$.

The corresponding closed-loop state matrix in the controller canonical realization appears to be

$$A = \begin{pmatrix} \frac{1}{1 + K_D} & -\frac{1}{1 + K_P} \\ K_I & \frac{1}{1 + K_D} \end{pmatrix}. $$

Using (5), it results

$$|M(k)| = \frac{4K_I(1 + K_P)^2}{(1 + K_D)^3},$$

which is a rational expression in accordance with the previous discussion. If a PI control would have been used instead (i.e. $K_D = 0$), then $|M(k)|$ would result to be the polynomial expression $4K_I(1 + K_P)^2$ in terms of the parameters $K_P$ and $K_I$.

Following the denotation in [2], the stability crossing boundaries are categorized as real-root-boundaries (RRB), complex-root-boundaries (CRB) and infinite-root-boundaries (IRB). They refer to the bounds in the control parameter space $k \in \mathbb{R}^p$ where some of the eigenvalues cross the origin, the imaginary axis or change the sign at infinity, respectively. As stated in the introduction, sweeping the frequency is inevitable in computing the CRB employing the existing parameter space methods [2] and [4]. Here, an alternative technique for calculating the stability crossing boundaries which dispense with the sweeping step is proposed. The key idea consists in computing the root crossing boundaries by checking the positive definiteness of the matrix $P(k)$ in the parameter space $\mathbb{R}^p$. In this regard, the parameter space regions need to be determined for which the determinants of the leading principal minors are positive.

To determine the control parameter values where the minors change the signs, their numerators and denominators are set equal to zero which requires the solution of a system of $2n$ equations with $k$ as the unknown. However, (3) infers that the denominators of all the entries of $P(k)$ are equal to $|M(k)|$ while the denominator of $|\mu_i(k)|$ in (2) equals $|M(k)|^i$. Clearly, $|M(k)|^i = 0$ has the same solution for $i = 1, \ldots, n$. Therefore, just $n+1$ independent equations are left to be solved which still demand a high computational effort.

However, it can be shown that the finite stability crossing boundaries are computed just by calculating the denominator of one of the leading principle minors and their numerators do not really play a role. It is demonstrated in the following, that the stability crossing boundaries RRB and CRB are the controller coefficients for which $|M(k)| = 0$ while the IRB bounds (if they exist) are computed from the condition $|M(k)| \rightarrow \infty$. To show this, recall that the eigenvalue-based stability condition is defined by checking the roots of the characteristic polynomial

$$|sI - A(k)| = 0. \quad (7)$$

Equivalently, the stability can be inferred by checking the positive definiteness of $P(k)$. In either case, the stability crossing boundaries are the controller coefficients for which the real parts of the eigenvalues of $A(k)$ or $P(k)$ change.
It turns out that the determinant of $M(k)$ in (5) is given based on [13] by

$$|M| = \prod_{i=1}^{n} \prod_{j=1}^{n} (\lambda_i + \lambda_j), \quad (8)$$

where $\lambda_i, \ldots, \lambda_n$ are the eigenvalues of $A$. If $s = 0$ satisfies (7), $A(k)$ has an eigenvalue at the origin, see Figure 1a and 1d (this referring to the RRB). Also, since $A(k)$ is real, if $s = j\omega$ satisfies (7), $A(k)$ has two conjugated eigenvalues $s = \pm j\omega$ on the imaginary axis, see Figure 1b and 1e (this referring to a CRB). For both cases, $|M(k)| = 0$ is concluded from ((8)). On the other side, if $|M(k)| = 0$, there exists either an eigenvalue $\lambda_i = 0$ or two imaginary eigenvalues $\lambda_i$ and $\lambda_j$ such that $\lambda_i = -\lambda_j$. Therefore, either $s = 0$ or $s = \pm j\omega$ satisfies (7). Hence, $s = 0$ and $s = j\omega$ satisfies (7), if and only if $|M(k)| = 0$ holds. Since, the $n$th principal minor of $P(k)$ is $|P(k)|$ and $|M(k)|$ is the denominator of $|P(k)|$, $|M(k)| = 0$ leads to $|P(k)| \to \infty$ which means at least one of the eigenvalues of $P$ goes to infinity. In this case, some eigenvalues of $P(k)$ change signs at infinity while some finite eigenvalues of $A(k)$ are changing signs by crossing the imaginary axis, see Figure 1c and 1f. As remarked, on the other hand, the IRB bounds exist only if $A(k)$ has at least one eigenvalue at infinity. Yet, (8) indicates that $s \to \infty$ satisfies (7), if and only if $|M| \to \infty$ holds. Since, $|M(k)| \to \infty$ leads to $|P(k)| = 0$, at least one eigenvalue of $P(k)$ must lie in the origin. Consider, finally, the controller coefficients are corresponding to the intersections of RRB and IRB or CRB and IRB. In these intersection points, $|M(k)|$ involves a zero-by-zero division.

Hence, the value of $|M(k)|$ is ambiguous. At these values of controller coefficients, some eigenvalues of $P(k)$ traverse to the positive plane or the negative plane, crossing the origin or infinity.

To summarize, for the computation of the characteristic stability boundaries (CRB, RRB and IRB) it is sufficient to check the mapping of the following two conditions

$$|M(k)| = 0, \quad |M(k)| \to \infty, \quad (9)$$

in the parameter space $k$. The first one refers to the closed-loop system eigenvalues of $A(k)$ on the imaginary axis, i.e. the eigenvalues of the Lyapunov matrix $P(k)$ at infinity while the latter one describes the system eigenvalues at infinity, i.e. the zero eigenvalues of the Lyapunov matrix.

III. CASE-STUDY PID CONTROL

In this section, the Lyapunov based technique is demonstrated in the case of PID control design.

A. Finite stability boundaries

For a more intuitive understanding of the proposed method, the calculation of finite boundaries (RRB and CRB) is presented in the following example. The stabilizing parameter space is calculated for a second order system

$$G_1(s) = \frac{K}{T^2s^2 + 2dT + 1}. \quad (10)$$

Matrix $A$ of the closed loop system results in

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\
-\frac{KK_t}{T^2} & -\frac{1+KK_p}{T^2} & -\frac{2d+KK_d}{T^2} \end{pmatrix}. $$

$$\text{(d) Eigenvalues of } A \text{ at the RRB}
\text{(e) Eigenvalues of } A \text{ at the CRB}
\text{(f) Eigenvalues of } A \text{ at the IRB}

Fig. 1: Eigenvalues of $A$ and $P$ at the root boundaries
The variables $K$, $T = 1$ and $d = -0.5$ are chosen as system variables. The negative damping factor results in an unstable second order system. $3 \times 3$ identity matrix is chosen for $Q$. In the given example $n > m + 1$ holds and $|M|$ is a pure polynomial expression, employing \( (5) \)

\[
\]

is computed. The RRB and CRB can be calculated from $|M| = 0$. In order to compare the results, the following boundaries are calculated by using the classical parameter space approach, discussed in [2], [3], [7].

\[
\text{RRB: } K_I = 0 \quad \text{CRB: } K_P = \frac{\omega^2 T^2 - 1}{K} \quad K_I = \frac{2dT}{K} + K_D \omega^2.
\]

Solving the CRB system of (11) leads to the result $K_I = -\left(1 + K_P\right) + \left(1 + K_P\right) K_D$. These are the same boundaries given by the Lyapunov based approach. As stated in [14] the phenomenon that the critical stability boundaries of $P$ are found in the denominator, have already been observed but the link to the boundary calculation was still missing.

In Figure 2 the stability boundaries and stable parameter spaces are presented for $K_P = 1$. The coloured lines belong to the stability boundaries where the green line corresponds to the RRB. The other curves are the stability critical as well as not stability critical boundaries CRBs. Therefore, one eigenvalue of $P$ jumps from negative to positive infinity at $K_I = 0$ while two eigenvalues jump at $K_I = 0$. This phenomenon corresponds to a jump at the boundary of the stable parameter space. If a proportional factor smaller than $K_P = -1$ is chosen, no stable area can be found. This corresponds to the result of calculating the stabilizing $K_P$-interval as presented in [3]. In the case $K_P = -2$ and constant $K_D = 3$ there are jumps within the eigenvalues from negative to positive infinity but they are in such a way that no point can be considered which is positive definite. The same holds for $K_P = -1$ (limit of the $K_P$ interval).

This is a special case where the boundaries given by (11) degenerate to one. The non-linear curves in Figure 2 are not stability critical boundaries. Also at these boundaries $|M|$ goes to infinity. However, due to that the fact that already some stability critical eigenvalue crossings for the $P$ matrix were happen, this additional eigenvalue crossings does not influence the stability of the system.

B. Infinite stability boundaries

The above mentioned example omits the infinite root boundary (IRB) because $n > m + 1$ holds in (10), $|M|$ will not have a denominator depending on the controller coefficients. Hence, an IRB will arise. In order to illustrate this, the feedback loop of the following plant

\[
G_2(s) = \frac{Ks}{T^2 s^2 + 2dT s + 1},
\]

and a PID controller is analysed. The closed loop system is of lower order than the example in Section III-A because one $s$ can be factorized in the characteristic equation. This results in the canonical form

\[
A_2 = \begin{bmatrix} 0 & \frac{1}{T^2 + K K_P} \\ \frac{1}{2dT + K K_P} & \frac{1}{T + K K_D} \end{bmatrix}.
\]

The resulting stability boundaries of the classical parameter space approach are easily verified

\[
\text{IRB: } K_D = -\frac{T^2}{K}, \quad \text{RRB: } K_I = -\frac{1}{K}.
\]

(12)

Using $K_P = 2$, $K$ and $T = 1$ as well as $d = -0.5$, the Lyapunov approach is used to calculate the stability region of the given system. The results can be seen in Figure 3. Clearly, the IRB of the system $K_D = -1$ is calculated (magenta line). As expected this boundary is not in the denominator of $P$ in (1). Looking at $M$ from (8),

\[
|M| = \frac{4(K_D - 1)}{(K_D + 3)^2},
\]

the RRB (green line) occurs in the numerator of $|M|$. The IRB occurs in the denominator of $|M|$. This results in the same two boundaries as in the classical approach. One eigenvalue of matrix $P$ crosses the origin for $K_I = 1$ at the IRB. This is contrary to the result of [14] in which the boundaries were found in the denominator of $P$ in (1). The red curves are not stability critical boundaries, similar to the previous case.

C. Non-minimum phase system

In this section, a high order non-minimum phase system is used to present the easy applicability of the proposed approach. Consider the following plant transfer function

\[
G_3(s) = \frac{-0.5s^4 - 7s^3 - 2s + 1}{s^7 + 11s^6 + 46s^5 + 95s^4 + 109s^3 + 74s^2 + 24s}.
\]
which is often used in the parameter space community because of the nicely shaped stability domain. A detailed discussion of this system can be found e.g. in [3]. In this example \( n = m + 1 \) holds. Hence, both numerator and denominator of \(|M|\) depend on the controller coefficients. Based on the Section II, the CRB and RRB can be computed from the numerator of \(|M|\) and the IRB from the denominator of \(|M|\). The non-trivial stability region in the \( K_I - K_D \) plane for \( K_P = 1 \) is presented in Figure 4. It displays two disconnected stability regions. for decreasing values of \( K_P \), this two areas are merging to one. As it could be seen in Figure 4, the number of resulting not stability critical stability boundaries increases for increasing system dimensions.

IV. CONCLUSION

The parameter space approach represents an intuitive method for the control design. Typically, it amounts to the computation of the stable regions in the controller parameter spaces. In this paper, an extension was proposed which is based on mapping the standard Lyapunov equation into the controller parameter space. Frequency sweeping has been mandatory in existing parameter space design approaches because closed loop poles may cross the imaginary axis at any frequency. In contrast to this, in our Lyapunov based technique such a gridding step is omitted because it is sufficient to check for the positive definiteness of the Lyapunov matrix \( P \) only. It turns out that the stabilizing regions in the controller parameter space are identified by exploiting a simple determinant condition which directly relates to the eigenvalues of the Lyapunov matrix \( P \). Moreover, we believe that the discussed approach is quite general and promises extensions to a variety of control problem settings in parameter spaces. Pressing research topics in this sense are extension to systems with time delay, MIMO-systems, discrete-time systems, etc.

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