LTM

Conserving Integrators for Constrained Mechanical Systems

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Two Equations of Motion in Continuous and in Discrete Form

In analytical mechanics, the analysis of energy has put forth two different kinds of equations of motion, being solved in different phase spaces. Let Q be a configuration manifold, then a LAGRANGIAN $L : TQ \to \mathbb{R}$ maps the tangent bundle to the real line. A variational principle leads to the well known **Euler-Lagrange equations**. A HAMILTONIAN $H : T^*Q \to \mathbb{R}$ on the cotangent bundle is obtained from L via the Legendre transformation and (if this transformation is invertible) **Hamilton's equations** are equivalent to the Euler-Lagrange equations.

Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L(\mathbf{q},\dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L(\mathbf{q},\dot{\mathbf{q}})}{\partial \mathbf{q}} = 0$$

Hamilton's equations:

$$\begin{pmatrix} \dot{\mathbf{q}}(t) \\ \dot{\mathbf{p}}(t) \end{pmatrix} = \mathbf{X}_H\left(\mathbf{q}(t),\mathbf{p}(t)\right) = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix} \mathrm{d}H\left(\mathbf{q}(t),\mathbf{p}(t)\right) = \begin{pmatrix} \frac{\partial H}{\partial \mathbf{q}} \\ \frac{\partial H}{\partial \mathbf{p}} \end{pmatrix}$$

Adept ways of discretizing these equations for algorithmic time-stepping inherit different conservation properties from the solutions of the underlying continuous system. A *symplectic-momentum integrator* is derived from a discrete variational principle, whereas discretizing Hamilton's equations by using discrete derivatives (introduced by O. Gonzalez) gives rise to an *energy-momentum scheme*.

Discrete Euler-Lagrange Equations:

$$\mathbf{D}_2\mathbb{L}(\mathbf{q}_{k+1},\mathbf{q}_k) + \mathbf{D}_1\mathbb{L}(\mathbf{q}_k,\mathbf{q}_{k-1}) = 0 \quad \forall k \in \mathbb{N}$$

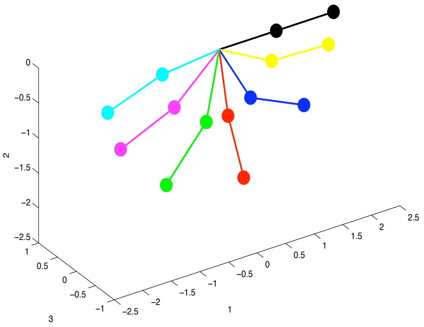
Discrete Hamilton's equations:

$$\begin{pmatrix} \mathbf{q}_{k+1} - \mathbf{q}_k \\ \mathbf{p}_{k+1} - \mathbf{p}_k \end{pmatrix} = h \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix} \mathrm{D}H\left((\mathbf{q}_k,\mathbf{p}_k),(\mathbf{q}_{k+1},\mathbf{p}_{k+1})\right) \quad \forall k \in \mathbb{N}$$

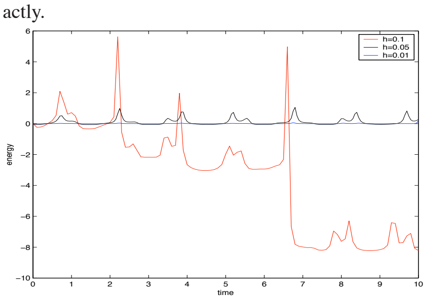
Example of a Constrained Mechanical System: The Double Spherical Pendulum

The motion of the double spherical pendulum is constrained by the constant length of the rods, which can be expressed by the condition $\mathbf{g}(\mathbf{q}) = \mathbf{0}$. Alternative ways to treat these holonomic constraints are represented either by adding different scalar valued functions $P(\mathbf{g}(\mathbf{q}))$ to the usual Lagrangian $L(\mathbf{q},\dot{\mathbf{q}}) = K(\dot{\mathbf{q}}) - U(\mathbf{q})$ and Hamiltonian $H(\mathbf{q},\mathbf{p}) = T(\mathbf{p}) + U(\mathbf{q})$ (being composed of kinetic energy K or T and potential energy U), respectively, or by coordinate transformation.

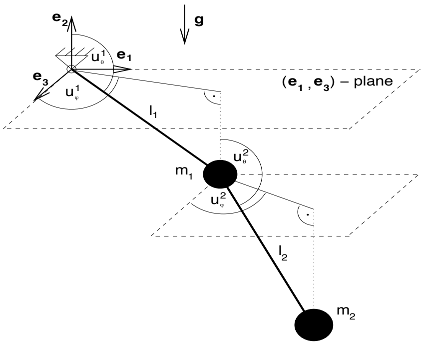
Lagrange Multipliers



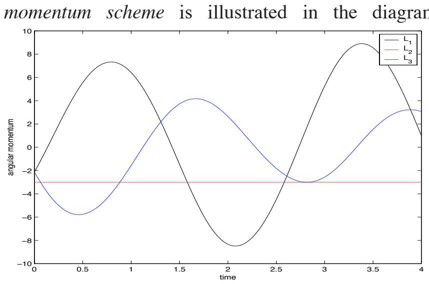
Approximation of the motion of the pendulum, starting at a horizontal initial position (black), by a *symplectic-momentum integrator* and treating the constraints by means of *Lagrange Multipliers* requires the addition of $P(\mathbf{g}(\mathbf{q})) = \lambda\,\mathbf{g}(\mathbf{q})$ to the LAGRANGIAN. In the *symplectic-momentum scheme* the total energy is not conserved, but the amplitude of the fluctuations decreases as the time step h decreases. The constraints are fulfilled exactly.



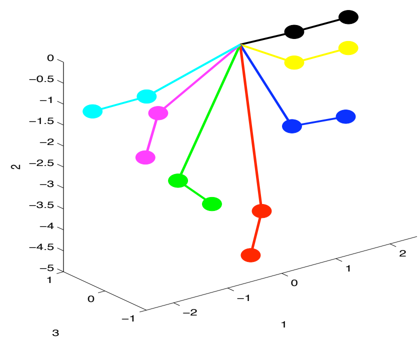
Generalized Coordinates



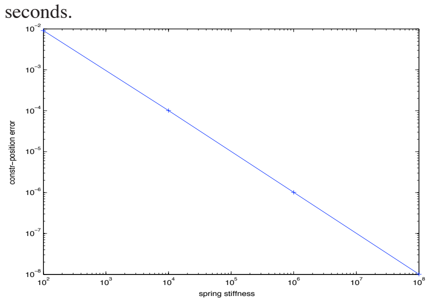
The reduction of the number of degrees of freedom by the constraints can be seen explicitely in the use of 2 spherical coordinates instead of 3 cartesian coordinates for the position of each mass. The system to be solved consists of less but more complicated equations including the transformation $\psi(\mathbf{u}) = \mathbf{q}$ and its derivatives. The conservation of the component of the angular momentum belonging to the gravitational axis along the motion calculated by the *symplectic-momentum scheme* is illustrated in the diagramm.



Penalty Potential



The constraint in the first rod is replaced by a spring with stiffness μ, introducing the penalty potential $P(\mathbf{g}(\mathbf{q})) = \mu||\mathbf{g}(\mathbf{q})||$. In the picture above, the variation in the length of the first rod during the motion of the pendulum can be seen for a spring of stiffness $\mu = 1$. The decrease in the error of the position constraints, as the spring stiffness increases, is depicted below for the *energy-momentum scheme* for a time step $h = 0.001$ seconds.



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