Efficient and robust integration of constrained dynamical systems

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Discrete null space method

Simulation algorithms are required to be efficient and robust while yielding accurate solutions that share the relevant properties of the continuous dynamical system, e.g. energy and momentum conservation and symplecticity. An elegant way to deduce such time stepping schemes is to parallel the continuous theory. The discrete null space method for the integration of constrained dynamical systems (BETSCH, LEYENDECKER) is based on this idea. To circumvent the presence of highly nonlinear transformations between the redundant coordinates $q \in \mathbb{R}^n$ (subject to holonomic constraints $g(q) \in \mathbb{R}^m$) and the independent generalised coordinates $u \in \mathbb{R}^{n-m}$ of the constraint manifold C during time discretisation, a system of differential-algebraic equations (DAEs) is discretised in time. However, the resulting time-stepping schemes has a relatively large dimension (it has to be solved for redundant coordinates plus Lagrange multipliers) and it may suffer from conditioning problems. Both drawbacks are eliminated by the application of the discrete null space method. Its main ingredients are a discrete null space matrix and a reparametrisation of the discrete configuration variable in independent incremental unknowns. After the discretisation of the DAEs has been completed, a size-reduction of the discrete system is performed by eliminating the constraint forces and introducing a reparametrisation of the discrete configuration variable. In the wake of the size-reduction, potential conditioning problems are eliminated, increasing the robustness of the simulation. The resulting time stepping scheme has the minimal possible dimension what benefits the simulations efficiency.

differential-algebraic equations of motion	energy-momentum scheme	symplectic-momentum scheme	o symplectic-energy- momentum scheme
$egin{array}{lll} M \cdot \ddot{q} + abla V(q) + G^T(q) \cdot \lambda &= 0 \ g(q) &= 0 \end{array}$	$\boldsymbol{M} \cdot [\boldsymbol{v}_{n+1} - \boldsymbol{v}_n] + \Delta t D V(\boldsymbol{q}_n, \boldsymbol{q}_{n+1}) + \Delta t \mathbf{G}^T(\boldsymbol{q}_n, \boldsymbol{q}_{n+1}) \cdot \boldsymbol{\lambda}_{n+1} = 0$ $\boldsymbol{g}(\boldsymbol{q}_{n+1}) = 0$	$D_1 L_d(q_n, q_{n+1}) + D_2 L_d(q_{n-1}, q_n) + G^T(q_n) \cdot \lambda_{n+1} = 0$ $g(q_{n+1}) = 0$	
$ \begin{array}{ll} \odot & {\pmb G}({\pmb q}) = D{\pmb g}({\pmb q}) \text{ constraint Jacobian} \\ \oplus & \text{constant mass matrix } {\pmb M} \\ \ominus & n+m \text{-dimensional system of index three} \end{array} $	$ \begin{aligned} & \odot \mathbf{G}(\boldsymbol{q}_n, \boldsymbol{q}_{n+1}) = D\boldsymbol{g}(\boldsymbol{q}_n, \boldsymbol{q}_{n+1}) \text{ discrete derivative (GONZALEZ)} \\ & \ominus n+m \text{-dimensional system} \\ & \ominus \text{condition number of iteration matrix is } \mathcal{O}(1/\Delta t^3) \end{aligned} $	\bigcirc discrete Lagrangian <i>L_d</i> ⊖ <i>n</i> + <i>m</i> -dimensional system $⊖$ condition number of iteration matrix is $O(1/\Delta t^3)$	 optimisation and control theory (DMOC)
null space matrix	discrete null space matrix	discrete null space matrix	
$\operatorname{range}\left(\boldsymbol{P}(\boldsymbol{q})\right) = \operatorname{null}\left(\boldsymbol{G}(\boldsymbol{q})\right)$	$\text{range}\left(\mathbf{P}(\boldsymbol{q}_n, \boldsymbol{q}_{n+1})\right) = \text{null}\left(\mathbf{G}(\boldsymbol{q}_n, \boldsymbol{q}_{n+1})\right)$	$\text{range}\left(\boldsymbol{P}(\boldsymbol{q}_n)\right) = \text{null}\left(\boldsymbol{G}(\boldsymbol{q}_n)\right)$	○ hierachical approach to
reparametrisation	discrete reparametrisation	discrete reparametrisation	large systems
$oldsymbol{q} = oldsymbol{F}(oldsymbol{u}) \in \mathcal{C}$ $oldsymbol{P}(oldsymbol{q}) = Doldsymbol{F}(oldsymbol{u})$ reduced equations of motion	$oldsymbol{q}_{n+1} = oldsymbol{F}_{n+1}(oldsymbol{u}) \in \mathcal{C}$	$oldsymbol{q}_{n+1} = oldsymbol{F}_{n+1}(oldsymbol{u}) \in \mathcal{C}$	
$DF^{T}(\boldsymbol{u}) \cdot \left[\boldsymbol{M} \cdot \frac{d}{dt} (DF(\boldsymbol{u}) \cdot \dot{\boldsymbol{u}}) + \nabla V(\boldsymbol{q}) \right] = \boldsymbol{0}$	reduced em-scheme $P^T(q_n, F_{n+1}(u)) \cdot [M \cdot [v_{n+1} - v_n] + \Delta t D V(q_n, F_{n+1}(u))] = 0$	reduced sm-scheme $\mathcal{P}^T(\boldsymbol{q}_n) \cdot \left[D_1 L_d(\boldsymbol{q}_n, \boldsymbol{F}_{n+1}(\boldsymbol{u})) + D_2 L_d(\boldsymbol{q}_{n-1}, \boldsymbol{q}_n) ight] ~=~ 0$	• separation of fast and slow changing degrees

- \oplus *n m* -dimensional system of second order ODEs
- \ominus **F** not always feasible
- ⊖ system is highly nonlinear

- \oplus *n m* -dimensional system
- \oplus condition number of iteration matrix is independent of Δt
- $\ominus \mathbf{P}^{T}(\boldsymbol{q}_{n}, \boldsymbol{q}_{n+1})$ depends on \boldsymbol{q}_{n+1}

- \oplus *n m* -dimensional system
- \oplus condition number of iteration matrix is independent of Δt
- $\oplus P^T(q_n)$ is independent of q_{n+1}

Extensions

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ıg. of freedom

Multibody system dynamics

six-body linkage: six rigid tetrahedra coupled by revolute joints, 1 degree of freedom

elastic slider-crank mechanism: two elastic beams (rigidly connected) and two rigid bodies (connected to beams by spherical joints), 214 degree of freedom

