Discrete Variational Mechanics for Elastic Rods

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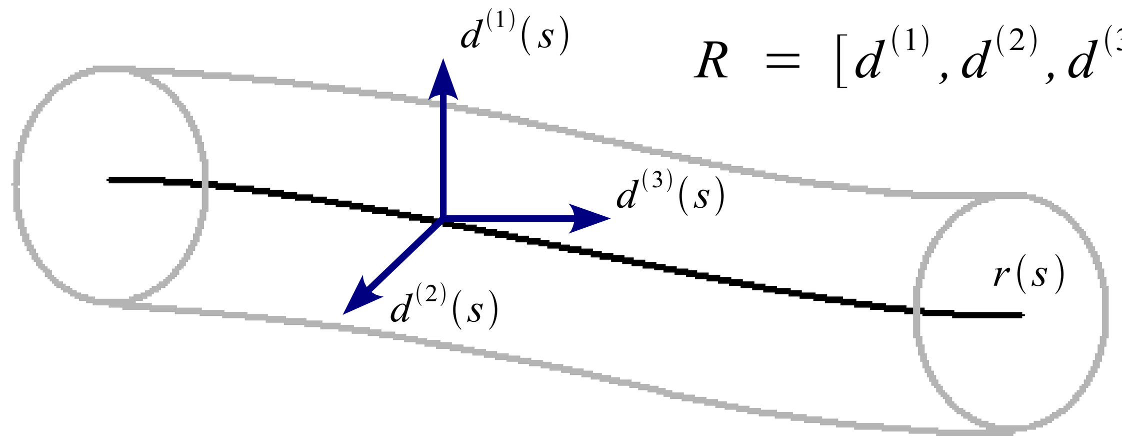
theory of special Cosserat rods

- deformed configuration (r, R)

$$r:[0,L] \to \mathbb{R}^3$$
$$R = [d^{(1)}, d^{(2)}, d^{(3)}]:[0,L] \to SO(3)$$



- strain vectors

$$\bar{u} = \left(d^{(3)}\cdot\frac{\partial}{\partial s}d^{(2)},\ d^{(1)}\cdot\frac{\partial}{\partial s}d^{(3)},\ d^{(2)}\cdot\frac{\partial}{\partial s}d^{(1)}\right), \qquad \bar{v} = R^{-1}\frac{\partial}{\partial s}r$$

flexure ↑ ↑ twist ↑ shear, dilatation ↑

material description : $\bar{u} = u_1 e_1 + u_2 e_2 + u_3 e_3$ ↔ spatial description : $u = u_1 d^{(1)} + u_2 d^{(2)} + u_3 d^{(3)}$

- hyperelastic rod: forces and moments are given by

$$\bar{m} = \frac{\partial w}{\partial \bar{u}}, \quad \bar{n} = \frac{\partial w}{\partial \bar{v}}$$

w: energy density function of a uniform rod, depending only on the strain vectors

- **idea:** rod problem (elastostatics) corresponds to a Lagrangian system on $\mathbb{R}^3 \times SO(3)$

$$W_{\text{elastic}} = \int_0^L w(\bar{u}(s), \bar{v}(s))\,\mathrm{d}s$$

$$\delta\left(\int_0^L w\ \mathrm{d}s\right) = 0$$

derive equilibrium equations from variational principle, handling the SO(3)-constraints with the null space method

- equilibrium equations
(spatial description)

$$\frac{\partial}{\partial s}n = 0$$
$$\frac{\partial}{\partial s}m + \frac{\partial}{\partial s}r \times n = 0$$

- momentum maps of frame-indifferent rods
(conserved quantities in space)
  - $n = const.$
  - $m + r \times n = const.$
  - $m \cdot n = \bar{m} \cdot \bar{n} = const.$
  - $m \cdot d^{(3)} = \bar{m} \cdot e_3 = const.$ (isotropic case)

concepts and techniques

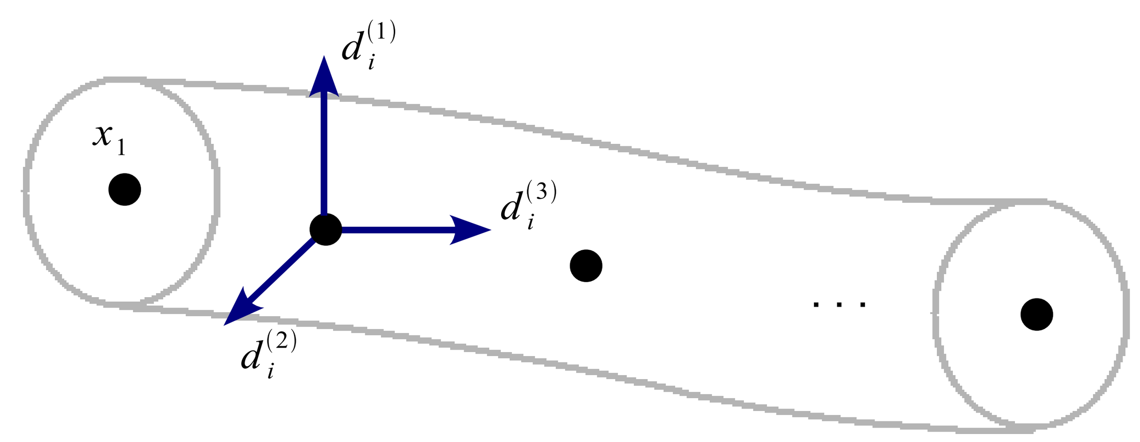
- discrete mechanics approach: switch to discrete setting at the earliest possible stage
- replace derivatives by finite differences
- apply the discrete null space method with nodal reparametrization

our rod model ...

- is not meant to replace more accurate FE models - but...
- displays all features of the continuum model qualitatively
- shows good accuracy with respect to the approximation of forces and moments
- computes fast and efficiently

discrete rod theory

(vertex-based discrete rod)



- definition of a discrete energy function

$$W_{\text{discrete}} = \sum_{i=1}^{N-1} W_i(q_i, q_{i+1}) \qquad (q_i \in \mathbb{R}^{12})$$

same concept as in variational integrator theory: construct numerical scheme from discrete variational principle

typical boundary data for rod problems:

$\{q_1, q_N\}$ i.e.
$\delta q_1 = \delta q_N = 0$

- 'spatial discrete Euler-Lagrange system'
(involving the boundary data q_1 and q_N)

$$\left|\begin{array}{l} P(q_i)^T\left(\dfrac{\partial W_{i-1}}{\partial q_i} + \dfrac{\partial W_i}{\partial q_i}\right) = 0 \\ g(q_i) = 0 \end{array}\right|$$

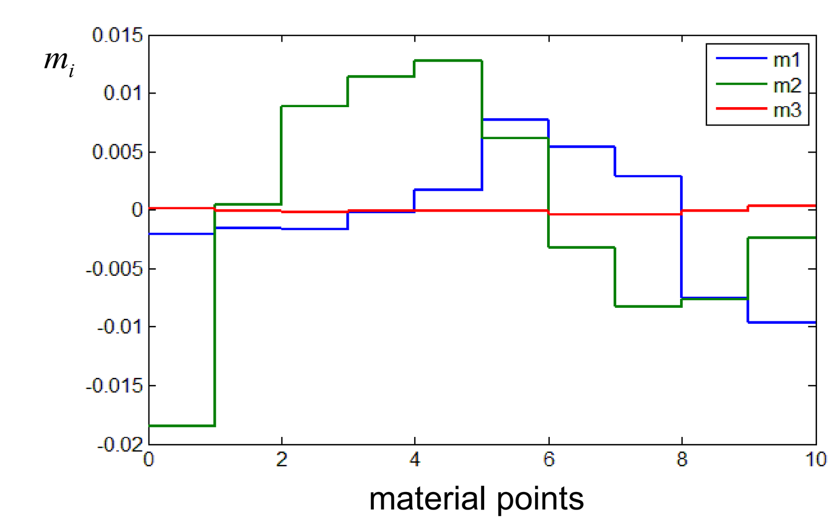
P: null space matrix, characterized by range $P(q) = \text{null}\ \mathrm{D}g(q)$

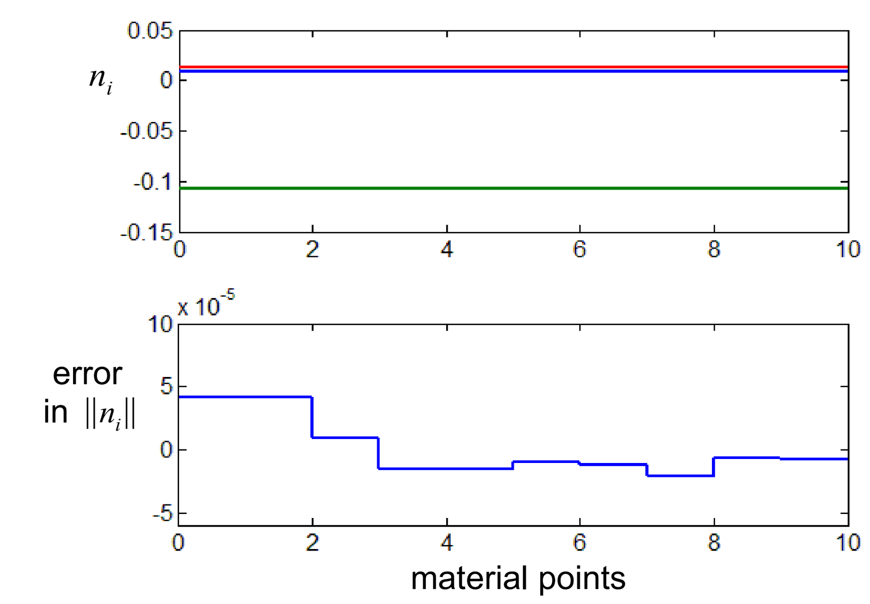
g: constraint function, enforcing the SO(3)-constraints

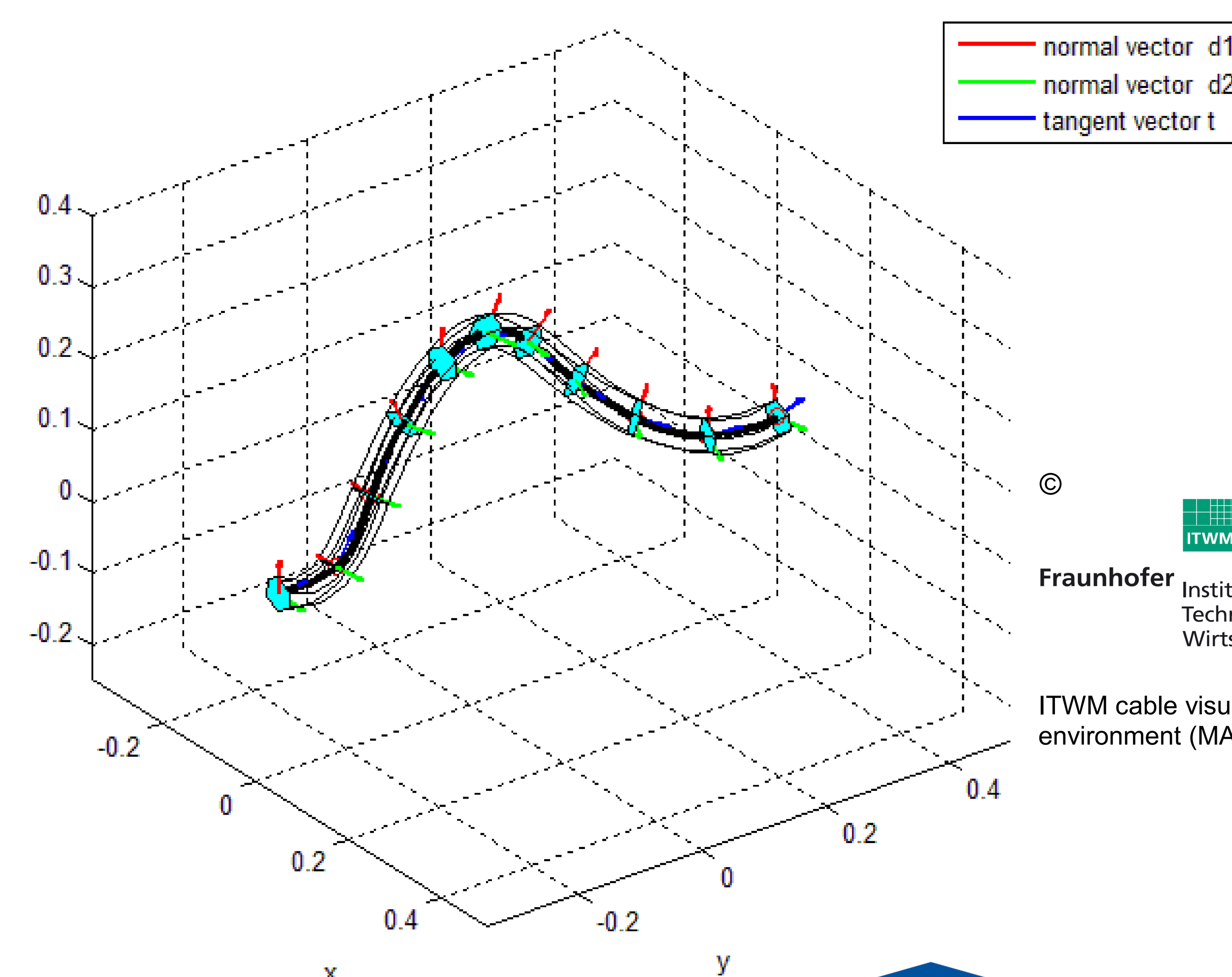
results

- nodal reparametrization allows to omit constraint function g
→ solve a system of minimum dimension
- fast computational performance
- exact fulfillment of constraints
- exact conservation of discrete momentum maps
(up to rootfinder precision)
- Kirchhof rod theory can easily be recovered by imposing additional constraints
- 'variational cable solver' is able to solve generalized boundary value problems (full clamping, free end, moment free and combinations thereof)

- **example:** forces and moments of a discrete rod







ITWM cable visualization environment (MATLAB)

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