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The discrete null space method for the energy consistent integration of constrained mechanical systems. Part II: Multibody dynamics

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SUMMARY

In the present work, rigid bodies and multibody systems are regarded as constrained mechanical systems at the outset. The constraints may be divided into two classes: (i) internal constraints which are intimately connected with the assumption of rigidity of the bodies, and (ii) external constraints related to the presence of joints in a multibody framework. Concerning external constraints lower kinematic pairs such as revolute and prismatic pairs are treated in detail. Both internal and external constraints are dealt with on an equal footing. The present approach thus circumvents the use of rotational variables throughout the whole time discretization. After the discretization has been completed a size-reduction of the discrete system is performed by eliminating the constraint forces. In the wake of the size-reduction potential conditioning problems are eliminated. The newly proposed methodology facilitates the design of energy–momentum methods for multibody dynamics. The numerical examples deal with a gyro top, cylindrical and planar pairs and a six-body linkage. Copyright © 2006 John Wiley & Sons, Ltd.

KEY WORDS: conserving time integration; constrained mechanical systems; multibody dynamics; differential-algebraic equations

1. INTRODUCTION

In the present work, we apply the discrete null space method developed by the first author [1] to rigid bodies and multibody systems. The precursor work [1] will be subsequently referred to as Part I.

The present formulation of multibody dynamics relies on redundant coordinates subject to holonomic constraints. In particular, as in Part I, the equations of motion assume the form of

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differential-algebraic equations (DAEs) with *constant* mass matrix. In this connection the most striking feature of the underlying rigid body description is the use of the nine components of the rotation matrix as redundant coordinates. At first glance this type of rigid body description seems to be quite awkward due to the large number of unknowns. For example, to describe purely rotational motion of a rigid body, nine coordinates together with six Lagrange multipliers for the enforcement of the constraints of rigidity are required. On the other hand, there are a number of reasons why the constrained formulation of rigid bodies becomes more and more popular, see, for example, the books by José and Saletan [2] and Leimkuhler and Reich [3].

The constrained formulation of rigid bodies is especially beneficial to the description of multibody systems. This is due to the fact that a general description of multibody systems has to take into account constraints anyway. In a multibody framework we may distinguish between internal constraints related to the rigidity of the individual bodies and external constraints associated with interconnecting joints. The underlying DAEs thus provide a uniform framework for both types of constraints and are thus well-suited for the description of multibody systems.

The constrained formulation of rigid bodies is also well-suited for structure preserving numerical time integration. See, for example, the symplectic scheme proposed by Reich [4] and the energy–momentum scheme proposed by Betsch and Steinmann [5]. Due to the uniform DAE formulation these time-stepping schemes can be directly extended to multibody dynamics. Indeed the design of energy–momentum methods for multibody dynamics within the framework of the discrete null space method is the main goal of the present work.

The more common description of rigid body dynamics by means of the Newton–Euler equations (see, for example, Reference [6]) may also serve as the starting point for the time discretization. In fact, a lot of work has been devoted to the design of structure preserving integrators for rigid bodies based on the classical Newton–Euler equations. In this connection, we refer to the papers by Lewis and Simo [7], Géradin and Rixen [8] and Krysl [9]. Conserving integrators for multibody dynamics have been developed in the framework of the Newton–Euler equations by Chen [10] and Lens *et al.* [11]. In a similar context the energy consistent discretization of joint constraints is addressed in References [12–14].

The present work aims at a systematic treatment of lower kinematic pairs within the discrete null space method developed in Part I. We make use of the aforementioned redundant coordinates for the description of the configuration of each individual rigid body. Consequently, the motion of rigid bodies, kinematic pairs as well as multibody systems is governed by a uniform set of DAEs with constant mass matrix. Similar to the internal constraints, the external constraints associated with the lower kinematic pairs considered herein are at most quadratic. These advantageous features facilitate a straight-forward energy–momentum conserving time integration. However, there are two major drawbacks of this approach: (i) an excessive number of unknowns (redundant coordinates plus Lagrange multipliers), and (ii) potential conditioning problems (cf. Part I). Both drawbacks can be completely eliminated by applying the discrete null space method. To achieve an efficient implementation of the discrete null space method the design of viable explicit representations of the discrete null space matrices is of primary importance. Accordingly, the design of null space matrices and its discrete counterparts for lower kinematic pairs is one of the main goals of the present work. To this end, we propose a multiplicative decomposition of the null space matrices which reflects the presence of internal and external constraints. In this connection, the external constraints are accounted for by incorporating the notion of a natural orthogonal complement introduced by Angeles and Lee [15].

This paper is organized as follows: Section 2 contains a short summary of the DAEs which uniformly govern the motion of the finite-dimensional mechanical systems considered herein. In addition to that, the discretization of the DAEs by means of the discrete null space method is outlined. Section 3 deals with the dynamics of a single rigid body. In this connection, both the DAE description and the application of the discrete null space method are treated. The extension to multibody dynamics is considered in Section 4. In particular, a systematic design of discrete null space matrices pertaining to lower kinematic pairs is proposed. Section 5 contains numerical examples dealing with a gyro top, cylindrical and planar pairs and a six-body linkage. Finally, conclusions are drawn in Section 6.

2. OUTLINE OF THE PRESENT APPROACH

This section provides an outline of the main ingredients of the present approach to the discretization of multibody dynamics. It essentially consists of the specific formulation of finite-dimensional constrained mechanical systems and the corresponding energy-consistent time discretization.

2.1. Constrained mechanical systems

We consider constrained mechanical systems which are governed by the following set of DAEs:

$$\begin{aligned} \dot{\mathbf{q}} - \mathbf{v} &= \mathbf{0} \\ \mathbf{M}\dot{\mathbf{v}} + \nabla V(\mathbf{q}) + \mathbf{G}^{\mathrm{T}}\boldsymbol{\lambda} &= \mathbf{0} \\ \boldsymbol{\Phi}(\mathbf{q}) &= \mathbf{0} \end{aligned} \tag{1}$$

where $\mathbf{q}(t) \in \mathbb{R}^n$ specifies the configuration of the mechanical system at time t. A superposed dot denotes differentiation with respect to time, $\mathbf{v} \in \mathbb{R}^n$ is the velocity vector and $\mathbf{M} \in \mathbb{R}^{n\times n}$ is a constant mass matrix, so that the kinetic energy can be written as

$$T(\mathbf{v}) = \tfrac{1}{2}\,\mathbf{v}^{\mathrm{T}}\mathbf{M}\mathbf{v} \tag{2}$$

Moreover, $V(\mathbf{q}) \in \mathbb{R}$ is a potential energy function, $\boldsymbol{\Phi}(\mathbf{q}) \in \mathbb{R}^m$ is a vector of geometric constraint functions, $\mathbf{G} = D\boldsymbol{\Phi}(\mathbf{q}) \in \mathbb{R}^{m\times n}$ is the constraint Jacobian and $\boldsymbol{\lambda} \in \mathbb{R}^m$ is a vector of multipliers which specify the relative magnitudes of the constraint forces. In the above description it is tacitly assumed that the m constraints at hand are independent.

Due to the presence of holonomic (or geometric) constraints $(1)_3$, the configuration manifold of the system is given by

$$Q = \{\mathbf{q}(t) \in \mathbb{R}^n \mid \boldsymbol{\Phi}(\mathbf{q}) = \mathbf{0}\} \tag{3}$$

The geometric constraints give rise to kinematic (or hidden) constraints which follow from the consistency condition $\mathrm{d}\boldsymbol{\Phi}/\mathrm{d}t = 0$. Accordingly, the kinematic constraints assume the form

$$\mathbf{G}\mathbf{v} = \mathbf{0} \tag{4}$$

Suppose that the columns of $\mathbf{P} \in \mathbb{R}^{n \times (n-m)}$ span the null space of $\mathbf{G} \in \mathbb{R}^{m \times n}$ and call $\mathbf{P}$ the null space matrix. Thus

$$\mathbf{G}\mathbf{P} = \mathbf{0} \tag{5}$$

and, consistent with (4), admissible velocities may be written in the form

$$\mathbf{v} = \mathbf{P}\boldsymbol{\nu} \tag{6}$$

with independent generalized velocities $\boldsymbol{\nu} \in \mathbb{R}^{n-m}$. These quantities may be classified as quasi-velocities because in general their time integrals need not result in generalized coordinates (cf. Reference [16]).

Using (6), the reduced form of the kinetic energy $\tilde{T}$ is defined by

$$\tilde{T}(\mathbf{q}, \boldsymbol{\nu}) = \frac{1}{2}\, \boldsymbol{\nu}^{\mathrm{T}} \widetilde{\mathbf{M}} \boldsymbol{\nu} \tag{7}$$

with the reduced mass matrix

$$\widetilde{\mathbf{M}} = \mathbf{P}^{\mathrm{T}} \mathbf{M} \mathbf{P} \tag{8}$$

Note that $\widetilde{\mathbf{M}}$ is generally configuration-dependent and assumed to be positive definite. The null space matrix can be employed to eliminate the forces of constraint. Specifically, premultiplying $(1)_2$ by $\mathbf{P}^{\mathrm{T}}$ and making use of (5) and (6) yields the alternative reduced formulation

$$\begin{aligned} \dot{\mathbf{q}} - \mathbf{P}\boldsymbol{\nu} &= \mathbf{0} \\ \widetilde{\mathbf{M}}\dot{\boldsymbol{\nu}} + \mathbf{P}^{\mathrm{T}}\mathbf{M}\dot{\mathbf{P}}\boldsymbol{\nu} + \mathbf{P}^{\mathrm{T}}\nabla V(\mathbf{q}) &= \mathbf{0} \\ \boldsymbol{\Phi}(\mathbf{q}) &= \mathbf{0} \end{aligned} \tag{9}$$

which governs the motion of the constrained mechanical system. In essence, the reduced formulation coincides with the d'Alembert-type formulation in Part I. A further size-reduction may be achieved by introducing appropriate 'generalized' coordinates for the parametrization of the configuration manifold Q.

2.2. Discrete null space method

The discrete null space method relies on the direct discretization of the constrained formulation (1), followed by a size-reduction of the nonlinear algebraic system to be solved. The discrete size-reduction can be viewed as discrete analogue of the continuous reduction procedure outlined above.

2.2.1. Direct discretization of the DAEs. In the present work, the DAEs (1) provide the underlying framework for the description of multibody systems. We choose to apply the following time discretization of the DAEs:

$$\begin{aligned} \mathbf{q}_{n+1} - \mathbf{q}_n &= \Delta t\, \mathbf{v}_{n+1/2} \\ \mathbf{M}[\mathbf{v}_{n+1} - \mathbf{v}_n] &= -\Delta t\, \bar{\nabla} V(\mathbf{q}_n, \mathbf{q}_{n+1}) - \Delta t\, \mathbf{G}(\mathbf{q}_n, \mathbf{q}_{n+1})^{\mathrm{T}} \bar{\boldsymbol{\lambda}} \\ \boldsymbol{\Phi}(\mathbf{q}_{n+1}) &= \mathbf{0} \end{aligned} \tag{10}$$

It is worthwhile noting that the constrained scheme (10) is energy consistent and conserves momentum maps associated with symmetries of the underlying mechanical system. Further details of the constrained scheme can be found in Part I of this work.

2.2.2. Discrete size-reduction

Discrete null space matrix: Similar to the continuous framework, the discrete null space matrix $\mathbf{P}(\mathbf{q}_n, \mathbf{q}_{n+1}) \in \mathbb{R}^{n\times(n-m)}$ is required for the elimination of the discrete multipliers $\bar{\boldsymbol{\lambda}} \in \mathbb{R}^m$. Specifically, we need to set up $\mathbf{P}(\mathbf{q}_n, \mathbf{q}_{n+1}) \in \mathbb{R}^{n\times(n-m)}$ such that,

$$\text{range}(\mathbf{P}(\mathbf{q}_n, \mathbf{q}_{n+1})) = \text{null}(\mathbf{G}(\mathbf{q}_n, \mathbf{q}_{n+1})) \tag{11}$$

Note that due to the properties of the discrete constraint Jacobian $\mathbf{G}(\mathbf{q}_n, \mathbf{q}_{n+1}) \in \mathbb{R}^{m\times n}$ (cf. Equation (18) in Part I), the mid-point velocities may be expressed as

$$\mathbf{v}_{n+1/2} = \mathbf{P}(\mathbf{q}_n, \mathbf{q}_{n+1})\mathbf{w} \tag{12}$$

with $\mathbf{w} \in \mathbb{R}^{n-m}$. The last equation can be interpreted as discrete version of (6). One major goal of this work is to find appropriate explicit representations of the discrete null space matrix $\mathbf{P}(\mathbf{q}_n, \mathbf{q}_{n+1}) \in \mathbb{R}^{n\times(n-m)}$ for multibody systems comprised of rigid bodies. In particular, the following two steps are proposed for the construction of viable discrete versions of the null space matrix:

Step 1: Set up an explicit representation of the continuous null space matrix $\mathbf{P}(\mathbf{q})$ by employing relationship (6).

Step 2: Find a proper discrete version $\mathbf{P}(\mathbf{q}_n, \mathbf{q}_{n+1})$ of the null space matrix such that the following design conditions are fulfilled:

(i) In the limit of vanishing time steps, $\Delta t \to 0$, the discrete version has to coincide with the continuous one. That is,

$$\mathbf{P}(\mathbf{q}_n, \mathbf{q}_{n+1}) \to \mathbf{P}(\mathbf{q}_n) \quad \text{as} \quad \mathbf{q}_{n+1} \to \mathbf{q}_n \tag{13}$$

(ii) To satisfy property (11), we further require that

$$\mathbf{G}(\mathbf{q}_n, \mathbf{q}_{n+1})\mathbf{P}(\mathbf{q}_n, \mathbf{q}_{n+1}) = \mathbf{0} \tag{14}$$

As it has been shown in Part I, due to property (11), the discrete multipliers can be eliminated from (10). The resulting scheme can be used to solve for $\mathbf{q}_{n+1} \in \mathbb{R}^n$ (see Section 3.2. in Part I).

Reparametrization of unknowns: If convergence of the solution procedure has been attained, $\mathbf{q}_{n+1} \in Q$, i.e. the solution belongs to the proper configuration manifold $Q \subset \mathbb{R}^n$. This feature of the constrained scheme (10) facilitates the introduction of incremental (generalized) coordinates $\mathbf{u} \in U \subset \mathbb{R}^{n-m}$ for the parametrization of the constraint manifold Q in the neighbourhood of $\mathbf{q}_n \in Q$. We thus introduce a mapping $\mathbf{F}_{q_n}\colon U \mapsto Q$, such that,

$$\mathbf{q}_{n+1} = \mathbf{F}_{q_n}(\mathbf{u}) \tag{15}$$

The final scheme can be applied to determine $\mathbf{u} \in \mathbb{R}^{n-m}$ (see Section 3.3. in Part I). We emphasize that the whole size-reduction of the discrete formulation does not affect the advantageous approximation properties of the underlying constrained scheme (cf. Section 3.4. in Part I).

3. RIGID BODIES

In the present work, we make use of a specific formulation of rigid bodies [5] which directly fits into the framework for constrained mechanical systems provided by the DAEs (1). The constrained formulation circumvents the need for using angular velocities and accelerations in the time discretization. Indeed the present discretization does not involve any type of rotational parameters. This is in contrast to time-stepping schemes based on the classical Euler's equations for rigid bodies (see, for example, References [9, 17, 18]).

The formulation of rigid bodies as discrete mechanical system subject to constraints appears to be especially beneficial to the description of multibody dynamics (cf. References [4, 19–22]). This is due to the fact that both 'internal' constraints associated with the rigidity of the bodies and 'external' constraints arising from the presence of joints can be dealt with on an equal footing.

The present constrained formulation of rigid bodies [5] relies on redundant absolute coordinates. Consequently, the (internal) constraints of rigidity are quadratic in the coordinates. Even the additional (external) constraints corresponding to the lower kinematic pairs dealt with in Section 4 turn out to be at most quadratic.

3.1. Constrained formulation of rigid body dynamics

The configuration of a rigid body in three-dimensional Euclidean space can be characterized by the placement of its centre of mass $\boldsymbol{\varphi}(t) \in \mathbb{R}^3$ and a right-handed body frame $\{\mathbf{d}_I\}$, $\mathbf{d}_I(t) \in \mathbb{R}^3$ $(I = 1, 2, 3)$, which specifies the orientation of the body (Figure 1). The vectors $\mathbf{d}_I$ will be occasionally called directors.

Let $\mathbf{X} = X_i \mathbf{e}_i$¶ be a material point which belongs to the reference configuration $V \subset \mathbb{R}^3$ of the rigid body. The spatial position of $\mathbf{X} \in V$ at time 't' relative to an inertial Cartesian basis $\{\mathbf{e}_I\}$ can now be characterized by

$$\mathbf{x}(\mathbf{X}, t) = \boldsymbol{\varphi}(t) + X_i \mathbf{d}_i(t) \tag{16}$$

For simplicity, we assume that the axes of the body frame are aligned with the principal axes of the body. Then the kinetic energy of the rigid body can be written as

$$T = \frac{1}{2} M_\varphi \|\mathbf{v}_\varphi\|^2 + \frac{1}{2} \sum_{I=1}^{3} E_I \|\mathbf{v}_I\|^2 \tag{17}$$

where $\mathbf{v}_\varphi = \dot{\boldsymbol{\varphi}}$, $\mathbf{v}_I = \dot{\mathbf{d}}_I$ and

$$\begin{aligned} M_\varphi &= \int_V \varrho(\mathbf{X})\, \mathrm{d}V \\ E_I &= \int_V (X_I)^2 \varrho(\mathbf{X})\, \mathrm{d}V \end{aligned} \tag{18}$$

¶In this work the summation convention applies to repeated lower case Roman indices.

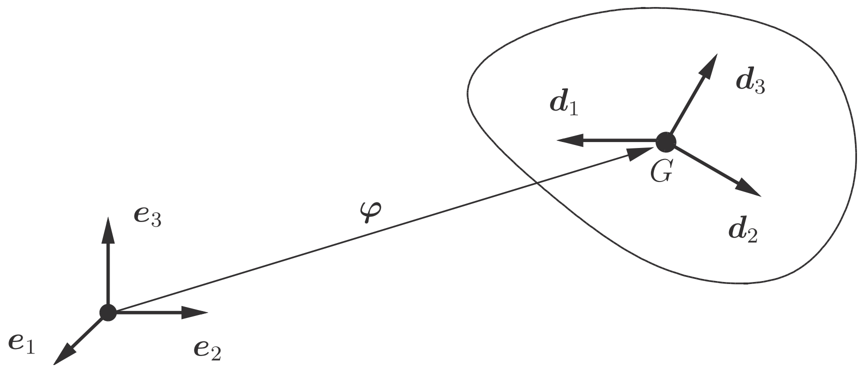

Figure 1. Configuration of a rigid body with respect to an orthonormal frame $\{\mathbf{e}_I\}$ fixed in space.

Here, $\varrho(\mathbf{X})$ is the mass density at $\mathbf{X} \in V$, M_φ is the total mass of the body and E_I are the principal values of the Euler tensor with respect to the centre of mass. Note that the spectral decomposition of the current Euler tensor with respect to the centre of mass is given by

$$\mathbf{E} = \sum_{I=1}^{3} E_I \mathbf{d}_I \otimes \mathbf{d}_I \tag{19}$$

The Euler tensor is symmetric positive definite, and can be linked to the customary inertia tensor via the relationship

$$\mathbf{J} = (\operatorname{tr} \mathbf{E})\mathbf{I} - \mathbf{E} \tag{20}$$

Obviously, the configuration of the rigid body can be characterized by the vector of redundant coordinates‖

$$\mathbf{q} = (\boldsymbol{\varphi}, \{\mathbf{d}_I\}) \in \mathbb{R}^3 \times \mathbb{R}^9 \cong \mathbb{R}^{12} \tag{21}$$

Correspondingly, the vector of redundant velocities may be written in the form

$$\mathbf{v} = (\mathbf{v}_\varphi, \{\mathbf{v}_I\}) \in \mathbb{R}^{12} \tag{22}$$

Expression (17) leads to the constant mass matrix

$$\mathbf{M} = \begin{bmatrix} M_\varphi \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & E_1 \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & E_2 \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & E_3 \mathbf{I} \end{bmatrix} \tag{23}$$

‖All vectors are to be formally regarded as column vectors although we sometimes abuse this convention for notational convenience.

where $\mathbf{I}$ and $\mathbf{0}$ are the 3×3 identity and zero matrices. Due to the assumption of rigidity, the body frame has to stay orthonormal for all times. Thus there are $m=6$ independent internal constraints with associated constraint functions.

$$\mathbf{\Phi}_{\text{int}}(\mathbf{q}) = \begin{bmatrix} \frac{1}{2}[\mathbf{d}_1^{\mathrm{T}}\mathbf{d}_1 - 1] \\ \frac{1}{2}[\mathbf{d}_2^{\mathrm{T}}\mathbf{d}_2 - 1] \\ \frac{1}{2}[\mathbf{d}_3^{\mathrm{T}}\mathbf{d}_3 - 1] \\ \mathbf{d}_1^{\mathrm{T}}\mathbf{d}_2 \\ \mathbf{d}_1^{\mathrm{T}}\mathbf{d}_3 \\ \mathbf{d}_2^{\mathrm{T}}\mathbf{d}_3 \end{bmatrix} \tag{24}$$

The internal constraints thus give rise to the corresponding 6×12 constraint Jacobian.

$$\mathbf{G}_{\text{int}}(\mathbf{q}) = \begin{bmatrix} \mathbf{0}^{\mathrm{T}} & \mathbf{d}_1^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & \mathbf{d}_2^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & \mathbf{d}_3^{\mathrm{T}} \\ \mathbf{0}^{\mathrm{T}} & \mathbf{d}_2^{\mathrm{T}} & \mathbf{d}_1^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0}^{\mathrm{T}} & \mathbf{d}_3^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & \mathbf{d}_1^{\mathrm{T}} \\ \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & \mathbf{d}_3^{\mathrm{T}} & \mathbf{d}_2^{\mathrm{T}} \end{bmatrix} \tag{25}$$

The equations of motion of the constrained system at hand can now be written in the form of the DAEs (1). Alternatively, one may apply the reduced form (9).

3.2. Reduced equations of motion for the rigid body

We next particularize the reduced equations of motion (9) for the rigid body. To this end an appropriate form of the null space matrix needs be found. This can be achieved by expressing the redundant velocities $\mathbf{v}\in\mathbb{R}^{12}$ of the (free) rigid body in terms of its twist. The twist is comprised of the angular velocity $\boldsymbol{\omega}\in\mathbb{R}^3$ and the translational velocity $\mathbf{v}_\varphi\in\mathbb{R}^3$ of the rigid body (see, for example, Reference [23]). It may be written as

$$\mathbf{t} = \begin{bmatrix} \mathbf{v}_\varphi \\ \boldsymbol{\omega} \end{bmatrix} \tag{26}$$

The director velocities $\mathbf{v}_I\in\mathbb{R}^3$ can now be expressed in terms of the angular velocity of the rigid body through

$$\mathbf{v}_I = \boldsymbol{\omega}\times\mathbf{d}_I = -\widehat{\mathbf{d}_I}\boldsymbol{\omega} \tag{27}$$

Here, $\widehat{\mathbf{a}}$ denotes the skew-symmetric 3×3 matrix with corresponding axial vector $\mathbf{a} \in \mathbb{R}^3$, that is

$$\widehat{\mathbf{a}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{28}$$

In view of (6), the components of the twist $\mathbf{t} \in \mathbb{R}^6$ play the role of independent generalized velocities for the free rigid body. Thus, we get the relationship $\mathbf{v} = \mathbf{P}_{\text{int}}\mathbf{t}$, which implies that the null space matrix for the free rigid body may be written as

$$\mathbf{P}_{\text{int}}(\mathbf{q}) = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\widehat{\mathbf{d}}_1 \\ \mathbf{0} & -\widehat{\mathbf{d}}_2 \\ \mathbf{0} & -\widehat{\mathbf{d}}_3 \end{bmatrix} \tag{29}$$

It can be easily verified that (i) the matrix in (29) has full column rank and, (ii) with regard to (25), $\mathbf{G}_{\text{int}}\mathbf{P}_{\text{int}} = \mathbf{0}$, the 6×6 zero matrix. The matrix in (29) thus qualifies perfectly as null space matrix.

The reduced mass matrix (8) pertaining to the rigid body can now be calculated by employing (23) and the null space matrix (29). Accordingly,

$$\widetilde{\mathbf{M}} = \mathbf{P}_{\text{int}}^{\text{T}}\mathbf{M}\mathbf{P}_{\text{int}} = \begin{bmatrix} M_\varphi \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\sum_{I=1}^{3} E_I (\widehat{\mathbf{d}}_I)^2 \end{bmatrix} \tag{30}$$

On the other hand,

$$\begin{aligned} -\sum_{I=1}^{3} E_I (\widehat{\mathbf{d}}_I)^2 &= -\sum_{I=1}^{3} E_I [\mathbf{d}_I \otimes \mathbf{d}_I - (\mathbf{d}_I^{\text{T}} \mathbf{d}_I)\mathbf{I}] \\ &= -\sum_{I=1}^{3} E_I \mathbf{d}_I \otimes \mathbf{d}_I + \sum_{I=1}^{3} E_I \mathbf{I} \end{aligned} \tag{31}$$

In the last equation, the property $\mathbf{d}_I^{\text{T}} \mathbf{d}_I = 1$ $(I = 1, 2, 3)$ has been incorporated, which follows from the geometric constraints. Taking into account $\operatorname{tr} \mathbf{E} = \sum_{I=1}^{3} E_I$ together with (20), the last equation yields

$$-\sum_{I=1}^{3} E_I (\widehat{\mathbf{d}}_I)^2 = \mathbf{J} \tag{32}$$

Accordingly, the reduced mass matrix of the rigid body can be written as

$$\widetilde{\mathbf{M}} = \begin{bmatrix} M_\varphi \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{J} \end{bmatrix} \tag{33}$$

In order to calculate the term $\mathbf{P}^{\mathrm{T}}\mathbf{M}\dot{\mathbf{P}}\mathbf{v}$ in $(9)_2$, we first perform the time derivative of the null space matrix which yields

$$\dot{\mathbf{P}}_{\mathrm{int}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\widehat{\mathbf{v}_1} \\ \mathbf{0} & -\widehat{\mathbf{v}_2} \\ \mathbf{0} & -\widehat{\mathbf{v}_3} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \widehat{\mathbf{d}_1 \times \boldsymbol{\omega}} \\ \mathbf{0} & \widehat{\mathbf{d}_2 \times \boldsymbol{\omega}} \\ \mathbf{0} & \widehat{\mathbf{d}_3 \times \boldsymbol{\omega}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\omega} \otimes \mathbf{d}_1 - \mathbf{d}_1 \otimes \boldsymbol{\omega} \\ \mathbf{0} & \boldsymbol{\omega} \otimes \mathbf{d}_2 - \mathbf{d}_2 \otimes \boldsymbol{\omega} \\ \mathbf{0} & \boldsymbol{\omega} \otimes \mathbf{d}_3 - \mathbf{d}_3 \otimes \boldsymbol{\omega} \end{bmatrix} \tag{34}$$

A straightforward calculation then gives the relationship

$$\mathbf{P}_{\mathrm{int}}^{\mathrm{T}} \mathbf{M} \dot{\mathbf{P}}_{\mathrm{int}} \mathbf{t} = \begin{bmatrix} \mathbf{0} \\ -\boldsymbol{\omega} \times \left(\sum_{I=1}^{3} E_I \mathbf{d}_I \otimes \mathbf{d}_I \right) \boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} \end{bmatrix} \tag{35}$$

where, (20) has been made use of. Finally, the last term in $(9)_2$ yields

$$\mathbf{P}_{\mathrm{int}}^{\mathrm{T}} \nabla V(\mathbf{q}) = \begin{bmatrix} \partial V / \partial \boldsymbol{\varphi} \\ \sum_{I=1}^{3} \mathbf{d}_I \times \partial V / \partial \mathbf{d}_I \end{bmatrix} =: - \begin{bmatrix} \bar{\mathbf{f}} \\ \bar{\mathbf{m}} \end{bmatrix} \tag{36}$$

where $\bar{\mathbf{f}}$ and $\bar{\mathbf{m}}$ is the resultant external force and torque relative to the centre of mass of the rigid body, respectively. To summarize, the reduced equations of motion in $(9)_2$ can be written in the familiar form,

$$\begin{aligned} M_{\varphi} \dot{\mathbf{v}}_{\varphi} &= \bar{\mathbf{f}} \\ \mathbf{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} &= \bar{\mathbf{m}} \end{aligned} \tag{37}$$

which represents the well-known Newton–Euler equations for rigid bodies.

3.3. Discrete null space method

The constrained scheme (10) can be directly applied to the present formulation of rigid body dynamics. In this connection, the discrete constraint Jacobian needs be specified. Since the internal constraints (24) are quadratic, the discrete constraint Jacobian coincides with the mid-point evaluation of the constraint Jacobian (25), i.e.

$$\mathbf{G}_{\mathrm{int}}(\mathbf{q}_n, \mathbf{q}_{n+1}) = \mathbf{G}_{\mathrm{int}}(\mathbf{q}_{n+1/2}) \tag{38}$$

The implementation of the constrained scheme (cf. Section 3.1.3. in Part I) for the free rigid body leads to a nonlinear system of algebraic equations in terms of $n+m=18$ unknowns. It is worth noting that the present discretization approach for rigid bodies (i) does not involve any rotational parameters, and (ii) yields a second-order accurate energy-momentum method (see also Reference [5]).

3.3.1. Discrete null space matrix. We next apply the two-step procedure outlined in Section 2.2.2 for the design of a discrete null space matrix pertaining to the free rigid body. Step 1 has already been performed and yields the null space matrix $\mathbf{P}_{\text{int}}(\mathbf{q})$ in (29). Then Step 2 gives rise to the discrete version of the null space matrix

$$\mathbf{P}_{\text{int}}(\mathbf{q}_n, \mathbf{q}_{n+1}) = \mathbf{P}_{\text{int}}(\mathbf{q}_{n+1/2}) \tag{39}$$

or

$$\mathbf{P}_{\text{int}}(\mathbf{q}_n, \mathbf{q}_{n+1}) = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -(\widehat{\mathbf{d}}_1)_{n+1/2} \\ \mathbf{0} & -(\widehat{\mathbf{d}}_2)_{n+1/2} \\ \mathbf{0} & -(\widehat{\mathbf{d}}_3)_{n+1/2} \end{bmatrix} \tag{40}$$

3.3.2. Reparametrization of the unknowns. Due to the six constraints of orthonormality (24), the configuration space of the free rigid body is

$$Q = \mathbb{R}^3 \times SO(3) \subset \mathbb{R}^3 \times \mathbb{R}^9 \tag{41}$$

where $SO(3)$ is the special orthogonal group. A reduction of the number of unknowns can now be achieved by introducing a rotation matrix $\mathbf{R}(\boldsymbol{\theta}) \in SO(3)$ parametrized in terms of $\boldsymbol{\theta} \in \mathbb{R}^3$, such that

$$(\mathbf{d}_I)_{n+1} = \mathbf{R}(\boldsymbol{\theta})(\mathbf{d}_I)_n \tag{42}$$

Thus, the three rotational variables $\boldsymbol{\theta} \in \mathbb{R}^3$ play the role of new incremental unknowns which can be used to express the original nine unknowns associated with the directors $(\mathbf{d}_I)_{n+1} \in \mathbb{R}^3$ $(I = 1, 2, 3)$. Concerning the rotation matrix, we choose to make use of the Rodrigues formula, which may be interpreted as closed-form expression of the exponential map (see, for example, Reference [24]):

$$\mathbf{R}(\boldsymbol{\theta}) = \exp(\widehat{\boldsymbol{\theta}}) = \mathbf{I} + \frac{\sin \|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|}\widehat{\boldsymbol{\theta}} + \frac{1}{2}\left(\frac{\sin(\|\boldsymbol{\theta}\|/2)}{\|\boldsymbol{\theta}\|/2}\right)^2 (\widehat{\boldsymbol{\theta}})^2 \tag{43}$$

When the above reparametrization of unknowns is applied, the configuration of the free rigid body is specified by six unknowns $\mathbf{u} = (\mathbf{u}_\varphi, \boldsymbol{\theta}) \in U \subset \mathbb{R}^3 \times \mathbb{R}^3$, characterizing the incremental displacement and the incremental rotation, respectively. Accordingly, in the present case the mapping $\mathbf{F}_{q_n}\colon U \mapsto Q$ assumes the form

$$\mathbf{q}_{n+1} = \mathbf{F}_{q_n}(\mathbf{u}) = \begin{bmatrix} \boldsymbol{\varphi}_n + \mathbf{u}_\varphi \\ \exp(\widehat{\boldsymbol{\theta}})(\mathbf{d}_1)_n \\ \exp(\widehat{\boldsymbol{\theta}})(\mathbf{d}_2)_n \\ \exp(\widehat{\boldsymbol{\theta}})(\mathbf{d}_3)_n \end{bmatrix} \tag{44}$$

We finally remark that the present use of rotation matrix (43) is restricted to a single time step such that possible singularities of (43) are not an issue in practical applications.

4. MULTIBODY SYSTEMS

We next illustrate the present approach to multibody systems by considering a system consisting of two rigid bodies interconnected by different types of joints. In particular, we consider two rigid links of a simple kinematic chain that are coupled by lower kinematic pairs.

4.1. Lower kinematic pairs

With regard to Section 3, the configuration of the αth rigid link in a kinematic chain can be characterized by redundant coordinates $\mathbf{q}^{\alpha} \in \mathbb{R}^{12}$. Thus, the configuration of two rigid links denoted by 1 and 2, can be characterized by 24 redundant coordinates, which may be arranged in the configuration vector

$$\mathbf{q} = \begin{bmatrix} \mathbf{q}^1 \\ \mathbf{q}^2 \end{bmatrix} \tag{45}$$

The constrained formulation of each rigid body leads to constraint functions $\mathbf{\Phi}_{\text{int}}^i \in \mathbb{R}^6$ of form (24) along with constraint Jacobians $\mathbf{G}_{\text{int}}^i$ of form (25). For the two-body system at hand, we thus get

$$\mathbf{\Phi}_{\text{int}}(\mathbf{q}) = \begin{bmatrix} \mathbf{\Phi}_{\text{int}}^1(\mathbf{q}^1) \\ \mathbf{\Phi}_{\text{int}}^2(\mathbf{q}^2) \end{bmatrix} \tag{46}$$

together with

$$\mathbf{G}_{\text{int}}(\mathbf{q}) = \begin{bmatrix} \mathbf{G}_{\text{int}}^1(\mathbf{q}^1) & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_{\text{int}}^2(\mathbf{q}^2) \end{bmatrix} \tag{47}$$

Accordingly, the two-body system under consideration leads to $m_{\text{int}} = 12$ internal constraints with associated constraint Jacobian $\mathbf{G}_{\text{int}} \in \mathbb{R}^{12\times 24}$. The coupling of the two bodies by means of a specific joint leads to further constraints, termed external constraints. Table I, contains a summary of lower kinematic pairs, $J \in \{R, P, C, S, E\}$, that will be investigated in the following. Depending on the number of external constraints, $m_{\text{ext}}^{(J)}$, the degrees of freedom (DOF) of the relative motion of one body with respect to the other is decreased. We refer to References [23, 25] for more background on kinematic pairs.

Each kinematic pair is characterized by altogether $m^{(J)} = m_{\text{int}} + m_{\text{ext}}^{(J)}$ constraints. The corresponding constraint functions may be arranged in the vector $\mathbf{\Phi}^{(J)} \in \mathbb{R}^{m^{(J)}}$, which may be written in partitioned form,

$$\mathbf{\Phi}^{(J)}(\mathbf{q}) = \begin{bmatrix} \mathbf{\Phi}_{\text{int}}(\mathbf{q}) \\ \mathbf{\Phi}_{\text{ext}}^{(J)}(\mathbf{q}) \end{bmatrix} \tag{48}$$

Table I. Lower kinematic pairs considered in the present work.

Kinematic pair (J)	Number of external constraints $m_{\text{ext}}^{(J)}$	DOF of the relative motion $r^{(J)}$	DOF of the pair $d^{(J)}$
Rotational (R)	5	1	7
Prismatic (P)	5	1	7
Cylindrical (C)	4	2	8
Spherical (S)	3	3	9
Planar (E)	3	3	9

Similarly, the constraint Jacobian $\mathbf{G}^{(J)} \in \mathbb{R}^{m^{(J)} \times 24}$ pertaining to a specific kinematic pair can be written as

$$\mathbf{G}^{(J)}(\mathbf{q}) = \begin{bmatrix} \mathbf{G}_{\text{int}}(\mathbf{q}) \\ \mathbf{G}_{\text{ext}}^{(J)}(\mathbf{q}) \end{bmatrix} \tag{49}$$

The equations of motion of each kinematic pair assume the form of the DAEs (1). In this connection, the constant mass matrix $\mathbf{M} \in \mathbb{R}^{24 \times 24}$ is given by

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}^1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^2 \end{bmatrix} \tag{50}$$

where each submatrix $\mathbf{M}^\alpha \in \mathbb{R}^{12 \times 12}$ coincides with (23).

4.2. Design of continuous null space matrices

In this section, we outline the construction of continuous null space matrices (Step 1 of the procedure in Section 2.2.2) for the kinematic pairs under consideration. Similar to the case of a single rigid body treated in Section 3.2, we introduce the twist of the kinematic pair,

$$\mathbf{t} = \begin{bmatrix} \mathbf{t}^1 \\ \mathbf{t}^2 \end{bmatrix} \tag{51}$$

where, analogous to (26), the twist of the αth body $\mathbf{t}^\alpha \in \mathbb{R}^6$, is given by

$$\mathbf{t}^\alpha = \begin{bmatrix} \mathbf{v}_\varphi^\alpha \\ \boldsymbol{\omega}^\alpha \end{bmatrix} \tag{52}$$

Now, similar to (29), the redundant velocities $\mathbf{v} = \dot{\mathbf{q}} \in \mathbb{R}^{24}$ of the kinematic pair may be expressed as

$$\mathbf{v} = \mathbf{P}_{\text{int}} \mathbf{t} \tag{53}$$

where the 24×12 matrix $\mathbf{P}_{\text{int}}$ is given by

$$\mathbf{P}_{\text{int}}(\mathbf{q}) = \begin{bmatrix} \mathbf{P}_{\text{int}}^1(\mathbf{q}^1) & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_{\text{int}}^2(\mathbf{q}^2) \end{bmatrix} \tag{54}$$

and $\mathbf{P}_{\text{int}}^{\alpha}$ is the null space matrix associated with the αth free body. With regard to (29) we have

$$\mathbf{P}_{\text{int}}^{\alpha} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\widehat{\mathbf{d}}_1^{\alpha} \\ \mathbf{0} & -\widehat{\mathbf{d}}_2^{\alpha} \\ \mathbf{0} & -\widehat{\mathbf{d}}_3^{\alpha} \end{bmatrix} \tag{55}$$

Note that, by design, $\mathbf{G}_{\text{int}}\mathbf{P}_{\text{int}} = \mathbf{0}$, the 12×12 zero matrix.

In a kinematic pair, $J \in \{R, P, C, S, E\}$, the interconnection of the two rigid bodies by means of a specific joint restricts the relative motion of the second body with respect to the first body (cf. Table I). The relative motion can be accounted for by introducing $r^{(J)} = 6 - m_{\text{ext}}^{(J)}$ joint velocities $\boldsymbol{\tau}^{(J)}$. Thus, the motion of the kinematic pair can be characterized by $d^{(J)} = 6 + r^{(J)}$ independent generalized velocities.

$$\mathbf{v}^{(J)} = \begin{bmatrix} \mathbf{t}^1 \\ \boldsymbol{\tau}^{(J)} \end{bmatrix} \tag{56}$$

In particular, introducing the $6 \times d^{(J)}$ matrix $\mathbf{P}_{\text{ext}}^{2,(J)}$, the twist of the second body $\mathbf{t}^2 \in \mathbb{R}^6$ can be expressed as

$$\boxed{\mathbf{t}^{2,(J)} = \mathbf{P}_{\text{ext}}^{2,(J)} \mathbf{v}^{(J)}} \tag{57}$$

Accordingly, the twist of the kinematic pair $J \in \{R, P, C, S, E\}$ can be written in the form,

$$\mathbf{t}^{(J)} = \mathbf{P}_{\text{ext}}^{(J)} \mathbf{v}^{(J)} \tag{58}$$

with the $12 \times d^{(J)}$ matrix $\mathbf{P}_{\text{ext}}^{(J)}$, which may be partitioned according to

$$\mathbf{P}_{\text{ext}}^{(J)} = \begin{bmatrix} \mathbf{I}_{6\times 6} & \mathbf{0}_{6\times r^{(J)}} \\ \multicolumn{2}{c}{\mathbf{P}_{\text{ext}}^{2,(J)}} \end{bmatrix} \tag{59}$$

Once $\mathbf{P}_{\text{ext}}^{(J)}$ has been established, the total null space matrix pertaining to the kinematic pair under consideration can be calculated from,

$$\mathbf{P}^{(J)} = \mathbf{P}_{\text{int}} \mathbf{P}_{\text{ext}}^{(J)} \tag{60}$$

or, in partitioned form,

$$\boxed{\mathbf{P}^{(J)} = \begin{bmatrix} \mathbf{P}_{\text{int}}^{1} & \mathbf{0}_{12\times r^{(J)}} \\ & \mathbf{P}_{\text{int}}^{2}\mathbf{P}_{\text{ext}}^{2,(J)} \end{bmatrix}} \tag{61}$$

The above procedure warrants the design of viable null space matrices which automatically satisfy the relationship,

$$\mathbf{G}^{(J)}\mathbf{P}^{(J)} = \mathbf{0} \tag{62}$$

To see this, calculate

$$\mathbf{G}^{(J)}\mathbf{P}^{(J)} = \begin{bmatrix} \mathbf{G}_{\text{int}}\mathbf{P}_{\text{int}}\mathbf{P}_{\text{ext}}^{(J)} \\ \widetilde{\mathbf{G}}_{\text{ext}}^{(J)}\mathbf{P}_{\text{ext}}^{(J)} \end{bmatrix} \tag{63}$$

where the $m_{\text{ext}}^{(J)} \times 12$ matrix $\widetilde{\mathbf{G}}_{\text{ext}}^{(J)}$ is given by

$$\widetilde{\mathbf{G}}_{\text{ext}}^{(J)} = \mathbf{G}_{\text{ext}}^{(J)}\mathbf{P}_{\text{int}} \tag{64}$$

Since, by construction, $\mathbf{G}_{\text{int}}\mathbf{P}_{\text{int}} = \mathbf{0}$, the upper block in (63) is equal to the $12 \times d^{(J)}$ zero matrix. In addition to that, the lower block in (63) is equal to the $m_{\text{ext}}^{(J)} \times d^{(J)}$ zero matrix, provided that $\mathbf{P}_{\text{ext}}^{2,(J)}$ has been properly deduced from (57).

To summarize, in order to construct a null space matrix pertaining to a specific kinematic pair, we essentially apply relationship (57) to deduce the matrix $\mathbf{P}_{\text{ext}}^{2,(J)}$. Once $\mathbf{P}_{\text{ext}}^{2,(J)}$ has been determined, the complete null space matrix pertaining to a specific pair follows directly from (61).

Remark 4.1
Similar to the above procedure for the design of null space matrices, the relationship between rigid body twists, and joint velocities is used in Reference [15] to deduce the 'natural orthogonal complement' in the context of simple kinematic chains comprised of elementary kinematic pairs.

4.3. Design of discrete null space matrices

We next apply Step 2 of the procedure in Section 2.2.2, to set up discrete null space matrices for the kinematic pairs under consideration. It is obvious from the above treatment of the single rigid body, that the discrete counterparts of $\mathbf{G}_{\text{int}}$ and $\mathbf{P}_{\text{int}}$ are given by

$$\begin{aligned} \mathbf{G}_{\text{int}}(\mathbf{q}_n, \mathbf{q}_{n+1}) &= \mathbf{G}_{\text{int}}(\mathbf{q}_{n+1/2}) \\ \mathbf{P}_{\text{int}}(\mathbf{q}_n, \mathbf{q}_{n+1}) &= \mathbf{P}_{\text{int}}(\mathbf{q}_{n+1/2}) \end{aligned} \tag{65}$$

It can be easily checked that,

$$\mathbf{G}_{\text{int}}(\mathbf{q}_{n+1/2})\mathbf{P}_{\text{int}}(\mathbf{q}_{n+1/2}) = \mathbf{0} \tag{66}$$

It is shown in Section 4.5, that all external constraint functions associated with the kinematic pairs under consideration are at most quadratic in the redundant coordinates. Consequently, due to the properties of the discrete derivative, the discrete constraint Jacobians are given by

$$\mathbf{G}_{\text{ext}}^{(J)}(\mathbf{q}_n, \mathbf{q}_{n+1}) = \mathbf{G}_{\text{ext}}^{(J)}(\mathbf{q}_{n+1/2}) \tag{67}$$

for $J \in \{R, P, C, S, E\}$.
Guided by the design of continuous null space matrices in the last section, we now set

$$\mathbf{P}^{(J)}(\mathbf{q}_n, \mathbf{q}_{n+1}) = \mathbf{P}_{\text{int}}(\mathbf{q}_{n+1/2})\mathbf{P}_{\text{ext}}^{(J)}(\mathbf{q}_n, \mathbf{q}_{n+1}) \tag{68}$$

where the discrete version of (59) is given by

$$\mathbf{P}_{\text{ext}}^{(J)}(\mathbf{q}_n, \mathbf{q}_{n+1}) = \begin{bmatrix} \mathbf{I}_{6\times 6} & \mathbf{0}_{6\times r^{(J)}} \\ \multicolumn{2}{c}{\mathbf{P}_{\text{ext}}^{2,(J)}(\mathbf{q}_n, \mathbf{q}_{n+1})} \end{bmatrix} \tag{69}$$

Note that $(65)_2$ has been accounted for in (68). Equation (68) together with (69), yields the discrete null space matrix pertaining to a specific kinematic pair.

$$\boxed{\mathbf{P}^{(J)}(\mathbf{q}_n, \mathbf{q}_{n+1}) = \begin{bmatrix} \mathbf{P}_{\text{int}}^{1}(\mathbf{q}_{n+1/2}^{1}) & \mathbf{0}_{12\times r^{(J)}} \\ \multicolumn{2}{c}{\mathbf{P}_{\text{int}}^{2}(\mathbf{q}_{n+1/2}^{2})\mathbf{P}_{\text{ext}}^{2,(J)}(\mathbf{q}_n, \mathbf{q}_{n+1})} \end{bmatrix}} \tag{70}$$

It thus remains to provide a discrete version of $\mathbf{P}_{\text{ext}}^{2,(J)}(\mathbf{q})$. To this end we now require the fulfillment of condition (14). In the present case, we get

$$\mathbf{G}^{(J)}(\mathbf{q}_{n+1/2})\mathbf{P}^{(J)}(\mathbf{q}_n, \mathbf{q}_{n+1}) = \begin{bmatrix} \mathbf{G}_{\text{int}}(\mathbf{q}_{n+1/2})\mathbf{P}_{\text{int}}(\mathbf{q}_{n+1/2})\mathbf{P}_{\text{ext}}^{(J)}(\mathbf{q}_n, \mathbf{q}_{n+1}) \\ \widetilde{\mathbf{G}}_{\text{ext}}^{(J)}(\mathbf{q}_{n+1/2})\mathbf{P}_{\text{ext}}^{(J)}(\mathbf{q}_n, \mathbf{q}_{n+1}) \end{bmatrix} \tag{71}$$

with

$$\widetilde{\mathbf{G}}_{\text{ext}}^{(J)}(\mathbf{q}_{n+1/2}) = \mathbf{G}_{\text{ext}}^{(J)}(\mathbf{q}_{n+1/2})\mathbf{P}_{\text{int}}(\mathbf{q}_{n+1/2}) \tag{72}$$

Note that in the above formulae (65) and (67) have been taken into account. We further, remark that (71) can be viewed as the discrete counterpart of (63). It follows from (72) that $\widetilde{\mathbf{G}}_{\text{ext}}^{(J)}(\mathbf{q}_{n+1/2})$ is prescribed for each kinematic pair $J \in \{R, P, C, S, E\}$. Substituting (66) into (71) yields,

$$\mathbf{G}^{(J)}(\mathbf{q}_{n+1/2})\mathbf{P}^{(J)}(\mathbf{q}_n, \mathbf{q}_{n+1}) = \begin{bmatrix} \mathbf{0} \\ \widetilde{\mathbf{G}}_{\text{ext}}^{(J)}(\mathbf{q}_{n+1/2})\mathbf{P}_{\text{ext}}^{(J)}(\mathbf{q}_n, \mathbf{q}_{n+1}) \end{bmatrix} \tag{73}$$

In order to complete Step 2 of the design procedure, we finally demand that

$$\begin{aligned} \mathbf{P}_{\text{ext}}^{(J)}(\mathbf{q}_n, \mathbf{q}_{n+1}) &\to \mathbf{P}_{\text{ext}}^{(J)}(\mathbf{q}_n) \quad \text{as } \mathbf{q}_{n+1} \to \mathbf{q}_n \\ \widetilde{\mathbf{G}}_{\text{ext}}^{(J)}(\mathbf{q}_{n+1/2})\mathbf{P}_{\text{ext}}^{(J)}(\mathbf{q}_n, \mathbf{q}_{n+1}) &= \mathbf{0} \end{aligned} \tag{74}$$

The design conditions in (74) can be used to determine a proper discrete version of $\mathbf{P}_{\text{ext}}^{(J)}$ (or, in view of (69), $\mathbf{P}_{\text{ext}}^{2,(J)}$). Then the discrete null space matrix pertaining to a specific kinematic pair follows from (70).

4.4. Reparametrization of unknowns

The redundant coordinates $\mathbf{q} \in \mathbb{R}^{24}$ of each kinematic pair $J \in \{R, P, C, S, E\}$ may be expressed, in terms of $d^{(J)} = 12 - m_{\text{ext}}^{(J)}$ independent generalized coordinates. Concerning, the reparametrization of unknowns in the discrete null space method, we seek for relationships of the form

$$\mathbf{q}_{n+1} = \mathbf{F}_{q_n}^{(J)}(\boldsymbol{\mu}^{(J)}) \tag{75}$$

where

$$\boldsymbol{\mu}^{(J)} = (\mathbf{u}_{\varphi}^1, \boldsymbol{\theta}^1, \boldsymbol{\vartheta}^{(J)}) \in \mathbb{R}^{d^{(J)}} \tag{76}$$

consists of a minimal number of incremental unknowns for a specific kinematic pair. In (76), $(\mathbf{u}_{\varphi}^1, \boldsymbol{\theta}^1) \in \mathbb{R}^3 \times \mathbb{R}^3$ are incremental displacements and rotations, respectively, associated with the first body (cf. Section 3.3.2). Furthermore, $\boldsymbol{\vartheta}^{(J)} \in \mathbb{R}^{r^{(J)}}$ are incremental unknowns, which characterize the configuration of the second body relative to the first one. In view of (45), the mapping in (75) may be partitioned according to

$$\mathbf{F}_{q_n}^{(J)}(\boldsymbol{\mu}^{(J)}) = \begin{bmatrix} \mathbf{F}_{q_n^1}^1(\mathbf{u}_{\varphi}^1, \boldsymbol{\theta}^1) \\ \mathbf{F}_{q_n}^{2,(J)}(\boldsymbol{\mu}^{(J)}) \end{bmatrix} \tag{77}$$

Alternatively, we may write

$$\begin{aligned} \mathbf{q}_{n+1}^1 &= \mathbf{F}_{q_n^1}^1(\mathbf{u}_{\varphi}^1, \boldsymbol{\theta}^1) \\ \mathbf{q}_{n+1}^2 &= \mathbf{F}_{q_n}^{2,(J)}(\boldsymbol{\mu}^{(J)}) \end{aligned} \tag{78}$$

Here, $\mathbf{F}_{q_n^1}^1(\mathbf{u}_{\varphi}^1, \boldsymbol{\theta}^1)$ is given by (44). It thus remains to specify the mapping $\mathbf{F}_{q_n}^{2,(J)}(\boldsymbol{\mu}^{(J)})$ for each kinematic pair under consideration. Of course, the mapping $\mathbf{F}_{q_n}^{(J)}$ is required to satisfy the constraints specified by (48), i.e. $\boldsymbol{\Phi}^{(J)}(\mathbf{F}_{q_n}^{(J)}) = \mathbf{0}$, for arbitrary $\boldsymbol{\mu}^{(J)}$.

4.5. Treatment of specific kinematic pairs

We next provide the details of the treatment of specific kinematic pairs $J \in \{R, P, C, S, E\}$. In essence, the present approach requires the specification of (i) the external constraint function $\boldsymbol{\Phi}_{\text{ext}}^{(J)}$, along with the corresponding constraint Jacobian $\mathbf{G}_{\text{ext}}^{(J)}$, and (ii) the null space matrix $\mathbf{P}_{\text{ext}}^{2,(J)}$, which is needed to set up the complete null space matrix (61) as described in Section 4.2. The corresponding discrete null space matrix is then deduced according to the procedure outlined in Section 4.3. Finally, (iii) the mapping $\mathbf{F}_{q_n}^{2,(J)}(\boldsymbol{\mu}^{(J)})$ is specified, which is needed to perform the reparametrization of unknowns according to (77), as described in Section 4.4.

In the following, we suppose that the location of a specific joint on each body is characterized by coordinates ϱ_i^α with respect to the respective body frame:

$$\boldsymbol{\varrho}^\alpha = \varrho_i^\alpha \mathbf{d}_i^\alpha \tag{79}$$

for $\alpha = 1, 2$.

4.5.1. Spherical pair

Constraints and constraint Jacobian: The S pair (Figure 2) entails three external constraints of the form

$$\mathbf{\Phi}_{\text{ext}}^{(S)}(\mathbf{q}) = \boldsymbol{\varphi}^2 - \boldsymbol{\varphi}^1 + \boldsymbol{\varrho}^2 - \boldsymbol{\varrho}^1 = \mathbf{0} \tag{80}$$

The corresponding constraint Jacobian is given by the constant 3×24 matrix.

$$\mathbf{G}_{\text{ext}}^{(S)}(\mathbf{q}) = [-\mathbf{I} \quad -\varrho_1^1 \mathbf{I} \quad -\varrho_2^1 \mathbf{I} \quad -\varrho_3^1 \mathbf{I} \quad \mathbf{I} \quad \varrho_1^2 \mathbf{I} \quad \varrho_2^2 \mathbf{I} \quad \varrho_3^2 \mathbf{I}] \tag{81}$$

Continuous form of the null space matrix: The motion of body 2 relative to body 1 is characterized by $r^{(S)} = 3$ DOF. Specifically, with regard to (56), we choose $\boldsymbol{\tau}^{(S)} = \boldsymbol{\omega}^2$, the angular velocity of the second body. Accordingly, in the present case, the vector of independent generalized velocities reads

$$\mathbf{v}^{(S)} = \begin{bmatrix} \mathbf{t}^1 \\ \boldsymbol{\omega}^2 \end{bmatrix} \tag{82}$$

Recall that the twist of the first rigid body is given by $\mathbf{t}^1 = (\mathbf{v}_\varphi^1, \boldsymbol{\omega}^1)$. Taking the time derivative of the external constraints (80) and expressing the redundant velocities in terms of the independent generalized velocities (82) yields,

$$\mathbf{v}_\varphi^2 = \mathbf{v}_\varphi^1 + \boldsymbol{\omega}^1 \times \boldsymbol{\varrho}^1 - \boldsymbol{\omega}^2 \times \boldsymbol{\varrho}^2 \tag{83}$$

Now it can be easily deduced from the relationship $\mathbf{t}^{2,(S)} = \mathbf{P}_{\text{ext}}^{2,(S)} \mathbf{v}^{(S)}$, that

$$\mathbf{P}_{\text{ext}}^{2,(S)}(\mathbf{q}) = \begin{bmatrix} \mathbf{I} & -\widehat{\boldsymbol{\varrho}^1} & \widehat{\boldsymbol{\varrho}^2} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{84}$$

so that (59) yields

$$\mathbf{P}_{\text{ext}}^{(S)} = \begin{bmatrix} \mathbf{I}_{6\times 6} & \mathbf{0}_{6\times 3} \\ \multicolumn{2}{c}{\mathbf{P}_{\text{ext}}^{2,(S)}} \end{bmatrix} \tag{85}$$

Furthermore, the null space matrix for the S pair follows directly from (61). It is given by

$$\mathbf{P}^{(S)}(\mathbf{q}) = \begin{bmatrix} \mathbf{P}_{\text{int}}^1 & \mathbf{0}_{6\times 3} \\ \multicolumn{2}{c}{\mathbf{P}_{\text{int}}^2 \mathbf{P}_{\text{ext}}^{2,(S)}} \end{bmatrix} \tag{86}$$

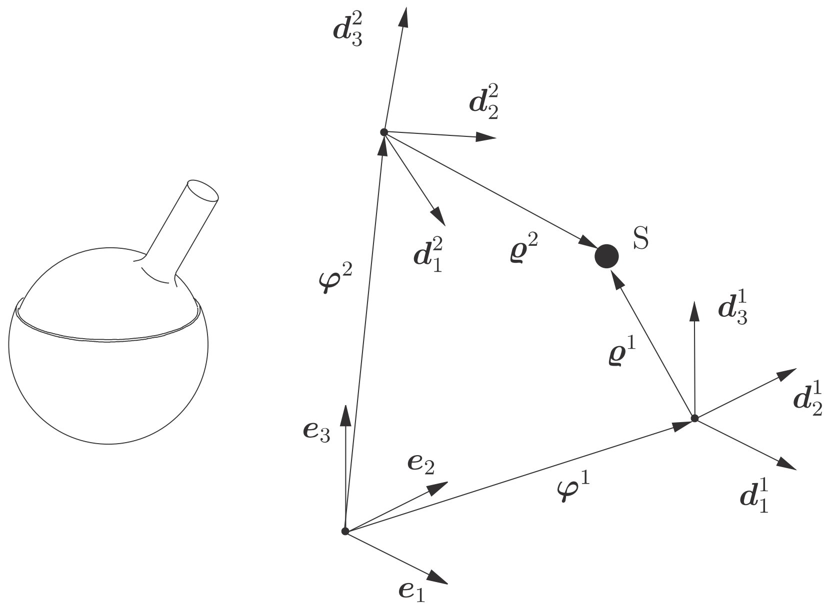

Figure 2. Spherical pair.

with

$$\mathbf{P}_{\text{int}}^{2}\mathbf{P}_{\text{ext}}^{2,(S)} = \begin{bmatrix} \mathbf{I} & -\widehat{\boldsymbol{\varrho}^1} & \widehat{\boldsymbol{\varrho}^2} \\ \mathbf{0} & \mathbf{0} & -\widehat{\mathbf{d}_1^2} \\ \mathbf{0} & \mathbf{0} & -\widehat{\mathbf{d}_2^2} \\ \mathbf{0} & \mathbf{0} & -\widehat{\mathbf{d}_3^2} \end{bmatrix} \tag{87}$$

It is worth mentioning that in the present case (64) reads

$$\widetilde{\mathbf{G}}_{\text{ext}}^{(S)} = \mathbf{G}_{\text{ext}}^{(S)}\mathbf{P}_{\text{int}} = [-\mathbf{I} \ \ \widehat{\boldsymbol{\varrho}^1} \ \ \mathbf{I} \ \ -\widehat{\boldsymbol{\varrho}^2}] \tag{88}$$

so that, as expected, the present design procedure for $\mathbf{P}_{\text{ext}}^{2,(S)}$ guarantees that

$$\widetilde{\mathbf{G}}_{\text{ext}}^{(S)}\mathbf{P}_{\text{ext}}^{(S)} = \mathbf{0} \tag{89}$$

Discrete version of the null space matrix: To set up, the discrete counterpart of the null space matrix $\mathbf{P}^{(S)}(\mathbf{q})$, we apply the procedure described in Section 4.3. To this end we choose

$$\mathbf{P}_{\text{ext}}^{2,(S)}(\mathbf{q}_n, \mathbf{q}_{n+1}) = \mathbf{P}_{\text{ext}}^{2,(S)}(\mathbf{q}_{n+1/2}) \tag{90}$$

It can be easily verified, that this choice fulfills the design conditions (74). In particular, it is obvious from (89) that

$$\widetilde{\mathbf{G}}_{\text{ext}}^{(S)}(\mathbf{q}_{n+1/2})\mathbf{P}_{\text{ext}}^{(S)}(\mathbf{q}_{n+1/2})=\mathbf{0} \tag{91}$$

Accordingly, we get

$$\mathbf{P}^{(S)}(\mathbf{q}_n,\mathbf{q}_{n+1})=\mathbf{P}^{(S)}(\mathbf{q}_{n+1/2}) \tag{92}$$

as discrete null space matrix for the S pair.

Reparametrization of unknowns: To specify, the reduced set of incremental unknowns (76) for the S pair, we choose $\boldsymbol{\vartheta}^{(S)}=\boldsymbol{\theta}^2\in\mathbb{R}^3$, the incremental rotation vector pertaining to the second body. Then the rotational update of the body frame associated with the second body can be performed according to

$$(\mathbf{d}_I^2)_{n+1}=\exp(\widehat{\boldsymbol{\theta}^2})(\mathbf{d}_I^2)_n \tag{93}$$

Enforcing the external constraints (80) at the end of the time step implies

$$\boldsymbol{\varphi}_{n+1}^2=\boldsymbol{\varphi}_{n+1}^1+\boldsymbol{\varrho}_{n+1}^1-\boldsymbol{\varrho}_{n+1}^2 \tag{94}$$

Eventually, the last two equations can be used to determine the mapping $\mathbf{F}_{q_n}^{2,(S)}(\boldsymbol{\mu}^{(S)})$.

$$\mathbf{q}_{n+1}^2=\mathbf{F}_{q_n}^{2,(S)}(\boldsymbol{\mu}^{(S)})=\begin{bmatrix}\boldsymbol{\varphi}_n^1+\mathbf{u}_\varphi^1+\exp(\widehat{\boldsymbol{\theta}^1})\boldsymbol{\varrho}_n^1-\exp(\widehat{\boldsymbol{\theta}^2})\boldsymbol{\varrho}_n^2\\ \exp(\widehat{\boldsymbol{\theta}^2})(\mathbf{d}_1^2)_n\\ \exp(\widehat{\boldsymbol{\theta}^2})(\mathbf{d}_2^2)_n\\ \exp(\widehat{\boldsymbol{\theta}^2})(\mathbf{d}_3^2)_n\end{bmatrix} \tag{95}$$

4.5.2. Cylindrical pair. For the C pair (Figure 3) we introduce a unit vector $\mathbf{n}^1$ which is fixed at the first body, and specified by constant components n_i^1 with respect to the body frame $\{\mathbf{d}_i^1\}$

$$\mathbf{n}^1=n_i^1\mathbf{d}_i^1 \tag{96}$$

In addition to that, we introduce two vectors

$$\mathbf{m}_\alpha^1=(m_\alpha^1)_i\mathbf{d}_i^1 \tag{97}$$

such that $\{\mathbf{m}_1^1,\mathbf{m}_2^1,\mathbf{n}^1\}$ constitute a right-handed orthonormal frame. The motion of the second body relative to the first one can be described by $r^{(C)}=2$ DOF. Translation along $\mathbf{n}^1$ and rotation about $\mathbf{n}^1$. The translational motion along $\mathbf{n}^1$ may be characterized by the displacement $u^2\in\mathbb{R}$, such that (see Figure 3)

$$\boldsymbol{\varphi}^1+\boldsymbol{\varrho}^1+u^2\mathbf{n}^1=\boldsymbol{\varphi}^2+\boldsymbol{\varrho}^2 \tag{98}$$

For the subsequent treatment, it proves convenient to introduce the vectors

$$\mathbf{p}^\alpha=\boldsymbol{\varphi}^\alpha+\boldsymbol{\varrho}^\alpha \tag{99}$$

for $\alpha=1,2$.

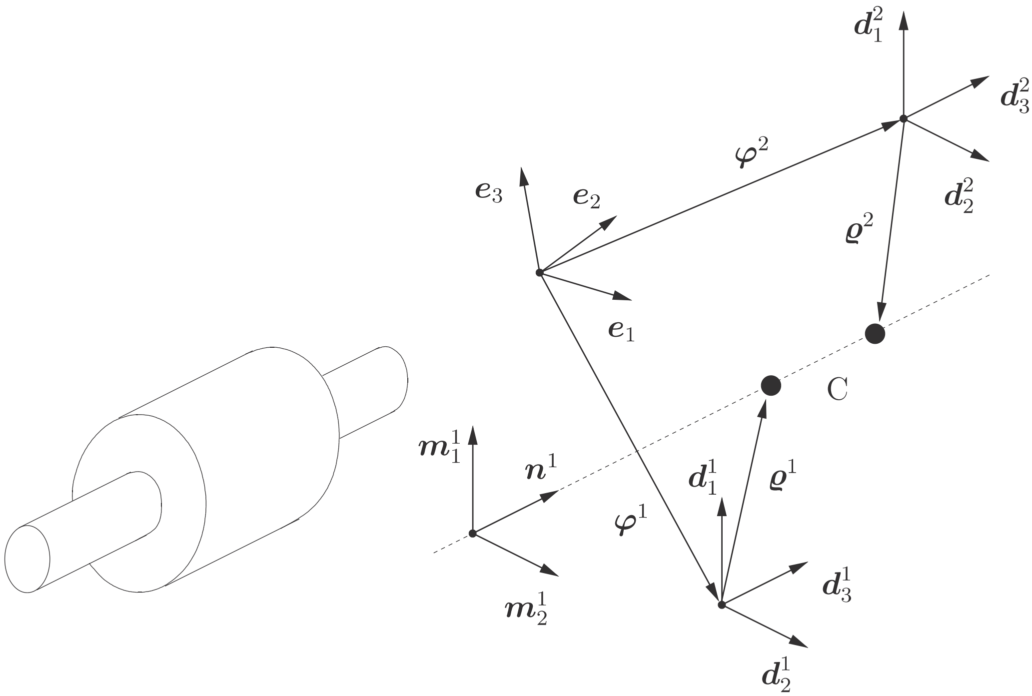

Figure 3. Cylindrical pair.

Constraints and constraint Jacobian: The C pair entails $m_{\text{ext}}^{(C)} = 4$ external constraint functions that may be written in the form

$$\boldsymbol{\Phi}_{\text{ext}}^{(C)}(\mathbf{q}) = \begin{bmatrix} (\mathbf{m}_1^1)^{\mathrm{T}}(\mathbf{p}^2 - \mathbf{p}^1) \\ (\mathbf{m}_2^1)^{\mathrm{T}}(\mathbf{p}^2 - \mathbf{p}^1) \\ (\mathbf{n}^1)^{\mathrm{T}}\mathbf{d}_1^2 - \eta_1 \\ (\mathbf{n}^1)^{\mathrm{T}}\mathbf{d}_2^2 - \eta_2 \end{bmatrix} \tag{100}$$

where η_1, η_2 are constant, and need to be consistent with the initial conditions. The first two components of (100) conform with (98) and thus, confine the translational motion of the second body relative to the first one. Similarly, the last two components of (100) restrict the relative rotational motion. The constraint Jacobian associated with (100) is given by the 4×24 matrix

$$\mathbf{G}_{\text{ext}}^{(C)}(\mathbf{q}) = \begin{bmatrix} -(\mathbf{m}_1^1)^{\mathrm{T}} & \mathbf{G}_{11}^{\mathrm{T}} & \mathbf{G}_{12}^{\mathrm{T}} & \mathbf{G}_{13}^{\mathrm{T}} & (\mathbf{m}_1^1)^{\mathrm{T}} & \varrho_1^2(\mathbf{m}_1^1)^{\mathrm{T}} & \varrho_2^2(\mathbf{m}_1^1)^{\mathrm{T}} & \varrho_3^2(\mathbf{m}_1^1)^{\mathrm{T}} \\ -(\mathbf{m}_2^1)^{\mathrm{T}} & \mathbf{G}_{21}^{\mathrm{T}} & \mathbf{G}_{22}^{\mathrm{T}} & \mathbf{G}_{23}^{\mathrm{T}} & (\mathbf{m}_2^1)^{\mathrm{T}} & \varrho_1^2(\mathbf{m}_2^1)^{\mathrm{T}} & \varrho_2^2(\mathbf{m}_2^1)^{\mathrm{T}} & \varrho_3^2(\mathbf{m}_2^1)^{\mathrm{T}} \\ \mathbf{0}^{\mathrm{T}} & n_1^1(\mathbf{d}_1^2)^{\mathrm{T}} & n_2^1(\mathbf{d}_1^2)^{\mathrm{T}} & n_3^1(\mathbf{d}_1^2)^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & (\mathbf{n}^1)^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0}^{\mathrm{T}} & n_1^1(\mathbf{d}_2^2)^{\mathrm{T}} & n_2^1(\mathbf{d}_2^2)^{\mathrm{T}} & n_3^1(\mathbf{d}_2^2)^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & (\mathbf{n}^1)^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} \end{bmatrix} \tag{101}$$

with

$$\mathbf{G}_{\alpha i} = (m^1_\alpha)_i(\mathbf{p}^2 - \mathbf{p}^1) - \varrho^1_i \mathbf{m}^1_\alpha \tag{102}$$

for $\alpha = 1, 2$ and $i = 1, 2, 3$.

Continuous form of the null space matrix: Corresponding to the $r^{(C)} = 2\,\mathrm{DOF}$, characterizing the motion of the second body relative to the first one, we choose

$$\boldsymbol{\tau}^{(C)} = \begin{bmatrix} \dot{u}^2 \\ \dot{\theta}^2 \end{bmatrix} \tag{103}$$

where, in addition to u^2 already introduced in (98), $\dot{\theta}^2$ accounts for the angular velocity of the second body relative to the first one. Specifically, we get

$$\boldsymbol{\omega}^2 = \boldsymbol{\omega}^1 + \dot{\theta}^2 \mathbf{n}^1 \tag{104}$$

The vector of independent generalized velocities pertaining to the C pair is now given by

$$\mathbf{v}^{(C)} = \begin{bmatrix} \mathbf{t}^1 \\ \dot{u}^2 \\ \dot{\theta}^2 \end{bmatrix} \tag{105}$$

Differentiating (98) with respect to time and taking into account (104) and (98), a straightforward calculation yields

$$\mathbf{v}^2_\varphi = \mathbf{v}^1_\varphi + \boldsymbol{\omega}^1 \times (\boldsymbol{\varphi}^2 - \boldsymbol{\varphi}^1) + \dot{u}^2 \mathbf{n}^1 + \dot{\theta}^2 \boldsymbol{\varrho}^2 \times \mathbf{n}^1 \tag{106}$$

Now, the twist of the second body can be expressed in terms of the independent generalized velocities via $\mathbf{t}^{2,(C)} = \mathbf{P}^{2,(C)}_{\mathrm{ext}} \mathbf{v}^{(C)}$, with the 6×8 matrix

$$\mathbf{P}^{2,(C)}_{\mathrm{ext}}(\mathbf{q}) = \begin{bmatrix} \mathbf{I} & \widehat{\boldsymbol{\varphi}^1 - \boldsymbol{\varphi}^2} & \mathbf{n}^1 & \boldsymbol{\varrho}^2 \times \mathbf{n}^1 \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{n}^1 \end{bmatrix} \tag{107}$$

Then (59) yields

$$\mathbf{P}^{(C)}_{\mathrm{ext}} = \begin{bmatrix} \mathbf{I}_{6\times 6} & \mathbf{0}_{6\times 2} \\ \multicolumn{2}{c}{\mathbf{P}^{2,(C)}_{\mathrm{ext}}} \end{bmatrix} \tag{108}$$

Finally, with regard to (61), the null space matrix for the C pair is given by

$$\mathbf{P}^{(C)}(\mathbf{q}) = \begin{bmatrix} \mathbf{P}^1_{\mathrm{int}} & \mathbf{0}_{6\times 2} \\ \multicolumn{2}{c}{\mathbf{P}^2_{\mathrm{int}} \mathbf{P}^{2,(C)}_{\mathrm{ext}}} \end{bmatrix} \tag{109}$$

with

$$\mathbf{P}_{\text{int}}^{2}\mathbf{P}_{\text{ext}}^{2,(C)}=\begin{bmatrix}\mathbf{I} & \widehat{\boldsymbol{\varphi}^1-\boldsymbol{\varphi}^2} & \mathbf{n}^1 & \boldsymbol{\varrho}^2\times\mathbf{n}^1\\ \mathbf{0} & -\widehat{\mathbf{d}_1^2} & \mathbf{0} & \mathbf{n}^1\times\mathbf{d}_1^2\\ \mathbf{0} & -\widehat{\mathbf{d}_2^2} & \mathbf{0} & \mathbf{n}^1\times\mathbf{d}_2^2\\ \mathbf{0} & -\widehat{\mathbf{d}_3^2} & \mathbf{0} & \mathbf{n}^1\times\mathbf{d}_3^2\end{bmatrix} \tag{110}$$

For later use, we calculate matrix (64), which in the present case is given by the 4×12 matrix

$$\widetilde{\mathbf{G}}_{\text{ext}}^{(C)}=\mathbf{G}_{\text{ext}}^{(C)}\mathbf{P}_{\text{int}}=\begin{bmatrix}-(\mathbf{m}_1^1)^{\mathrm{T}} & -\mathbf{G}_{1i}^{\mathrm{T}}\widehat{\mathbf{d}_i^1} & (\mathbf{m}_1^1)^{\mathrm{T}} & -(\mathbf{m}_1^1)^{\mathrm{T}}\widehat{\boldsymbol{\varrho}^2}\\ -(\mathbf{m}_2^1)^{\mathrm{T}} & -\mathbf{G}_{2i}^{\mathrm{T}}\widehat{\mathbf{d}_i^1} & (\mathbf{m}_2^1)^{\mathrm{T}} & -(\mathbf{m}_2^1)^{\mathrm{T}}\widehat{\boldsymbol{\varrho}^2}\\ \mathbf{0}^{\mathrm{T}} & -(\mathbf{d}_1^2)^{\mathrm{T}}\widehat{\mathbf{n}^1} & \mathbf{0}^{\mathrm{T}} & -(\mathbf{n}^1)^{\mathrm{T}}\widehat{\mathbf{d}_1^2}\\ \mathbf{0}^{\mathrm{T}} & -(\mathbf{d}_2^2)^{\mathrm{T}}\widehat{\mathbf{n}^1} & \mathbf{0}^{\mathrm{T}} & -(\mathbf{n}^1)^{\mathrm{T}}\widehat{\mathbf{d}_2^2}\end{bmatrix} \tag{111}$$

It can be easily checked by a straightforward calculation, that the present design procedure for $\mathbf{P}_{\text{ext}}^{2,(C)}$ ensures that

$$\widetilde{\mathbf{G}}_{\text{ext}}^{(C)}\mathbf{P}_{\text{ext}}^{(C)}=\mathbf{0} \tag{112}$$

Discrete version of the null space matrix: In the present case, the discrete null space matrix does not coincide with $\mathbf{P}^{(C)}(\mathbf{q}_{n+1/2})$. Instead, applying the procedure described in Section 4.3, we choose

$$\mathbf{P}_{\text{ext}}^{2,(C)}(\mathbf{q}_n,\mathbf{q}_{n+1}) = \begin{bmatrix}\mathbf{I} & \widehat{\boldsymbol{\varphi}_{n+1/2}^1-\boldsymbol{\varphi}_{n+1/2}^2} & (\mathbf{m}_1^1)_{n+1/2}\times(\mathbf{m}_2^1)_{n+1/2} & \boldsymbol{\varrho}_{n+1/2}^2\times\mathbf{n}_{n+1/2}^1\\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{n}_{n+1/2}^1\end{bmatrix} \tag{113}$$

In this connection, we remark that in general $(\mathbf{m}_1^1)_{n+1/2}\times(\mathbf{m}_2^1)_{n+1/2}$ does not coincide with $\mathbf{n}_{n+1/2}^1$, although $\mathbf{m}_1^1\times\mathbf{m}_2^1=\mathbf{n}_1^1$ in the continuous case. This is due to the fact that, in the discrete setting, the internal constraints of orthonormality of the $\mathbf{d}_i^1$'s are only enforced at the time nodes. In any case, it can be easily verified that (113) fulfills the design conditions (74). Finally, in view of (70), the discrete null space matrix for the C pair assumes the form

$$\mathbf{P}^{(C)}(\mathbf{q}_n,\mathbf{q}_{n+1})=\begin{bmatrix}\mathbf{P}_{\text{int}}^1(\mathbf{q}_{n+1/2}^1) & \mathbf{0}_{6\times 2}\\ \multicolumn{2}{c}{\mathbf{P}_{\text{int}}^2(\mathbf{q}_{n+1/2}^2)\mathbf{P}_{\text{ext}}^{2,(C)}(\mathbf{q}_n,\mathbf{q}_{n+1})}\end{bmatrix} \tag{114}$$

where

$$\mathbf{P}_{\text{int}}^{2}(\mathbf{q}_{n+1/2}^{2})\mathbf{P}_{\text{ext}}^{2,(C)}(\mathbf{q}_n,\mathbf{q}_{n+1})$$
$$=\begin{bmatrix} \mathbf{I} & \widehat{\boldsymbol{\varphi}_{n+1/2}^{1}-\boldsymbol{\varphi}_{n+1/2}^{2}} & (\mathbf{m}_1^1)_{n+1/2}\times(\mathbf{m}_2^1)_{n+1/2} & \boldsymbol{\varrho}_{n+1/2}^{2}\times\mathbf{n}_{n+1/2}^{1} \\ \mathbf{0} & -(\widehat{\mathbf{d}_1^2})_{n+1/2} & \mathbf{0} & \mathbf{n}_{n+1/2}^{1}\times(\mathbf{d}_1^2)_{n+1/2} \\ \mathbf{0} & -(\widehat{\mathbf{d}_2^2})_{n+1/2} & \mathbf{0} & \mathbf{n}_{n+1/2}^{1}\times(\mathbf{d}_2^2)_{n+1/2} \\ \mathbf{0} & -(\widehat{\mathbf{d}_3^2})_{n+1/2} & \mathbf{0} & \mathbf{n}_{n+1/2}^{1}\times(\mathbf{d}_3^2)_{n+1/2} \end{bmatrix} \tag{115}$$

Reparametrization of unknowns: For the C pair, the configuration of the second body with respect to the first one can be characterized by $\boldsymbol{\vartheta}^{(C)}=(u^2,\theta^2)\in\mathbb{R}^2$. Here, θ^2 accounts for the incremental relative rotation, which may be expressed via the product of exponentials formula

$$(\mathbf{d}_I^2)_{n+1}=\exp(\widehat{\boldsymbol{\theta}^1})\exp(\theta^2(\widehat{\mathbf{n}^1})_n)(\mathbf{d}_I^2)_n \tag{116}$$

Enforcing the external constraints (98) at the end of the time step implies

$$\boldsymbol{\varphi}_{n+1}^{2}=\boldsymbol{\varphi}_{n+1}^{1}+\boldsymbol{\varrho}_{n+1}^{1}-\boldsymbol{\varrho}_{n+1}^{2}+(u_n^2+u^2)\mathbf{n}_{n+1}^{1} \tag{117}$$

Accordingly, the mapping $\mathbf{F}_{q_n}^{2,(C)}(\boldsymbol{\mu}^{(C)})$ can be written in the form

$$\mathbf{q}_{n+1}^{2}=\mathbf{F}_{q_n}^{2,(C)}(\boldsymbol{\mu}^{(C)})=\begin{bmatrix} \boldsymbol{\varphi}_n^1+\mathbf{u}_\varphi^1+\exp(\widehat{\boldsymbol{\theta}^1})[\boldsymbol{\varrho}_n^1-\exp(\theta^2(\widehat{\mathbf{n}^1})_n)\boldsymbol{\varrho}_n^2+(u_n^2+u^2)\mathbf{n}_n^1] \\ \exp(\widehat{\boldsymbol{\theta}^1})\exp(\theta^2(\widehat{\mathbf{n}^1})_n)(\mathbf{d}_1^2)_n \\ \exp(\widehat{\boldsymbol{\theta}^1})\exp(\theta^2(\widehat{\mathbf{n}^1})_n)(\mathbf{d}_2^2)_n \\ \exp(\widehat{\boldsymbol{\theta}^1})\exp(\theta^2(\widehat{\mathbf{n}^1})_n)(\mathbf{d}_3^2)_n \end{bmatrix} \tag{118}$$

4.5.3. Revolute pair. As for the cylindrical pair, we make use of the unit vector $\mathbf{n}^1$ given by (96), which specifies the axis of rotation of the second body relative to the first one.

Constraints and constraint Jacobian: The R pair (Figure 4) entails $m_{\text{ext}}^{(R)}=5$ external constraint functions, which may be written in the form

$$\boldsymbol{\Phi}_{\text{ext}}^{(R)}(\mathbf{q})=\begin{bmatrix} \boldsymbol{\varphi}^2-\boldsymbol{\varphi}^1+\boldsymbol{\varrho}^2-\boldsymbol{\varrho}^1 \\ (\mathbf{n}^1)^{\mathrm{T}}\mathbf{d}_1^2-\eta_1 \\ (\mathbf{n}^1)^{\mathrm{T}}\mathbf{d}_2^2-\eta_2 \end{bmatrix} \tag{119}$$

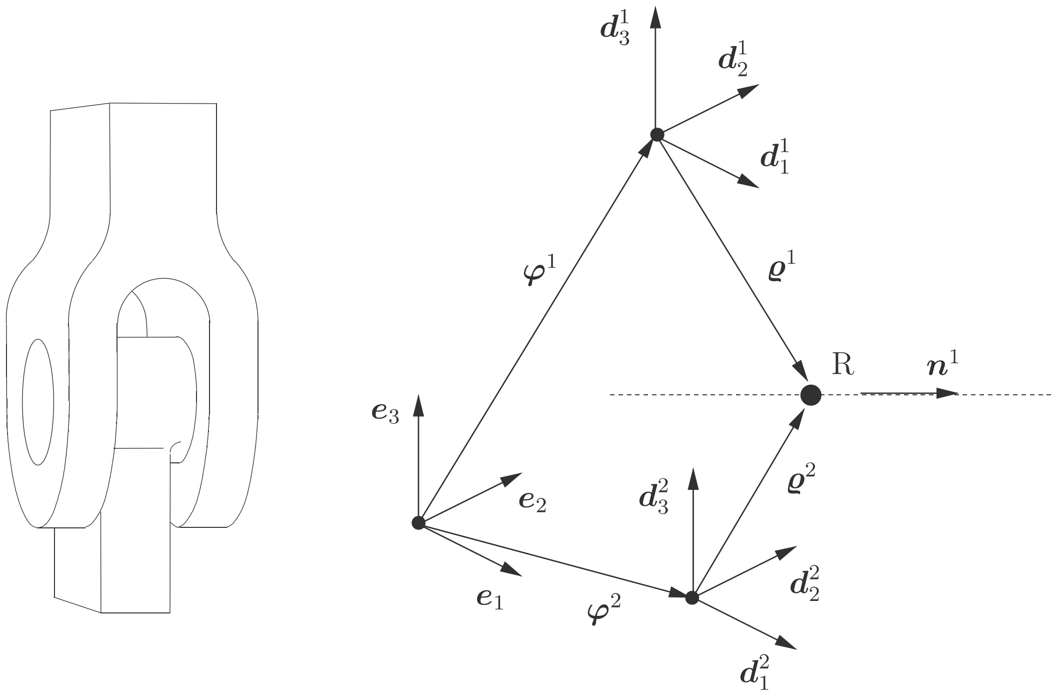

Figure 4. Revolute pair.

Analogous to the cylindrical pair η_1, η_2 are constant, and need to be consistent with the initial conditions. The corresponding constraint Jacobian is given by the 5×24 matrix

$$\mathbf{G}_{\text{ext}}^{(R)}(\mathbf{q}) = \begin{bmatrix} -\mathbf{I} & -\varrho_1^1\mathbf{I} & -\varrho_2^1\mathbf{I} & -\varrho_3^1\mathbf{I} & \mathbf{I} & \varrho_1^2\mathbf{I} & \varrho_2^2\mathbf{I} & \varrho_3^2\mathbf{I} \\ \mathbf{0}^{\text{T}} & n_1^1(\mathbf{d}_1^2)^{\text{T}} & n_2^1(\mathbf{d}_1^2)^{\text{T}} & n_3^1(\mathbf{d}_1^2)^{\text{T}} & \mathbf{0}^{\text{T}} & (\mathbf{n}^1)^{\text{T}} & \mathbf{0}^{\text{T}} & \mathbf{0}^{\text{T}} \\ \mathbf{0}^{\text{T}} & n_1^1(\mathbf{d}_2^2)^{\text{T}} & n_2^1(\mathbf{d}_2^2)^{\text{T}} & n_3^1(\mathbf{d}_2^2)^{\text{T}} & \mathbf{0}^{\text{T}} & \mathbf{0}^{\text{T}} & (\mathbf{n}^1)^{\text{T}} & \mathbf{0}^{\text{T}} \end{bmatrix} \tag{120}$$

Continuous and discrete form of the null space matrix: Both the continuous and the discrete null space matrix for the R pair can be directly inferred from the previous treatment of the cylindrical pair. Since the R pair does not allow purely translational motion of the second body relative to the first one, the corresponding column in the null space matrix (associated with $\dot{u}^2$) of the C pair has to be eliminated. This is consistent with the fact that the R pair has only one ($r^{(R)}=1$) DOF which characterizes the rotational motion of the second body relative to the first one. In particular, relationship (104) applies again. Note that, similar to (106), we now have

$$\mathbf{v}_\varphi^2 = \mathbf{v}_\varphi^1 + \boldsymbol{\omega}^1 \times (\boldsymbol{\varrho}^1 - \boldsymbol{\varrho}^2) + \dot{\theta}^2 \boldsymbol{\varrho}^2 \times \mathbf{n}^1 \tag{121}$$

which follows from differentiating with respect to time the first three constraint equations resulting from (119) and taking into account (104). Now, similar to (107), (121) gives rise to

$$\mathbf{P}_{\text{ext}}^{2,(R)}(\mathbf{q}) = \begin{bmatrix} \mathbf{I} & \widehat{\boldsymbol{\varrho}^2 - \boldsymbol{\varrho}^1} & \boldsymbol{\varrho}^2 \times \mathbf{n}^1 \\ \mathbf{0} & \mathbf{I} & \mathbf{n}^1 \end{bmatrix} \tag{122}$$

In this connection, we remark that the first three constraints resulting from (119), imply that $\boldsymbol{\varrho}^2 - \boldsymbol{\varrho}^1 = \boldsymbol{\varphi}^1 - \boldsymbol{\varphi}^2$. Proceeding along the lines of the previous treatment of the C pair, we now get

$$\mathbf{P}^{(R)}(\mathbf{q}) = \begin{bmatrix} \mathbf{P}_{\text{int}}^1 & \mathbf{0}_{6\times 1} \\ & \mathbf{P}_{\text{int}}^2 \mathbf{P}_{\text{ext}}^{2,(R)} \end{bmatrix} \tag{123}$$

with

$$\mathbf{P}_{\text{int}}^2 \mathbf{P}_{\text{ext}}^{2,(R)} = \begin{bmatrix} \mathbf{I} & \widehat{\boldsymbol{\varrho}^2 - \boldsymbol{\varrho}^1} & \boldsymbol{\varrho}^2 \times \mathbf{n}^1 \\ \mathbf{0} & -\widehat{\mathbf{d}_1^2} & \mathbf{n}^1 \times \mathbf{d}_1^2 \\ \mathbf{0} & -\widehat{\mathbf{d}_2^2} & \mathbf{n}^1 \times \mathbf{d}_2^2 \\ \mathbf{0} & -\widehat{\mathbf{d}_3^2} & \mathbf{n}^1 \times \mathbf{d}_3^2 \end{bmatrix} \tag{124}$$

In addition to that, the discrete null space matrix for the R pair follows from the mid-point evaluation of (123), that is,

$$\mathbf{P}^{(R)}(\mathbf{q}_n, \mathbf{q}_{n+1}) = \mathbf{P}^{(R)}(\mathbf{q}_{n+1/2}) \tag{125}$$

Reparametrization of unknowns: For the R pair, the mapping $\mathbf{F}_{q_n}^{2,(R)}(\boldsymbol{\mu}^{(R)})$ can be directly obtained from that of the C pair by fixing $u^2 = 0$. Then, the incremental rotational motion of the second body relative to the first one is specified by $\vartheta^{(R)} = \theta^2 \in \mathbb{R}$. With regard to (118), we thus get

$$\mathbf{q}_{n+1}^2 = \mathbf{F}_{q_n}^{2,(R)}(\boldsymbol{\mu}^{(R)}) = \begin{bmatrix} \boldsymbol{\varphi}_n^1 + \mathbf{u}_\varphi^1 + \exp(\widehat{\boldsymbol{\theta}^1})[\boldsymbol{\varrho}_n^1 - \exp(\theta^2(\widehat{\mathbf{n}^1})_n)\boldsymbol{\varrho}_n^2] \\ \exp(\widehat{\boldsymbol{\theta}^1})\exp(\theta^2(\widehat{\mathbf{n}^1})_n)(\mathbf{d}_1^2)_n \\ \exp(\widehat{\boldsymbol{\theta}^1})\exp(\theta^2(\widehat{\mathbf{n}^1})_n)(\mathbf{d}_2^2)_n \\ \exp(\widehat{\boldsymbol{\theta}^1})\exp(\theta^2(\widehat{\mathbf{n}^1})_n)(\mathbf{d}_3^2)_n \end{bmatrix} \tag{126}$$

4.5.4. Prismatic pair. In the case of the P pair (Figure 5), translational motion of the second body relative to the first one may occur along the axis specified by the unit vector $\mathbf{n}^1$, which, as before is specified by (96). Analogous to (98), we get the kinematic relationship

$$\boldsymbol{\varphi}^1 + \boldsymbol{\varrho}^1 + u^2\mathbf{n}^1 = \boldsymbol{\varphi}^2 + \boldsymbol{\varrho}^2 \tag{127}$$

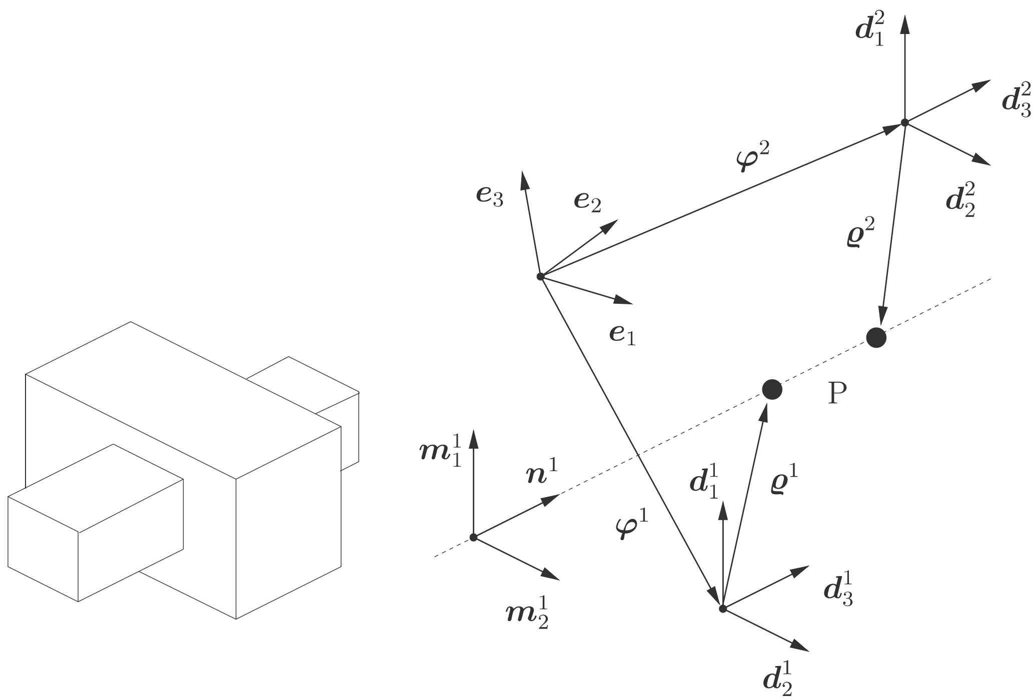

Figure 5. Prismatic pair.

Furthermore, the kinematic constraint

$$\boldsymbol{\omega}^2 = \boldsymbol{\omega}^1 \tag{128}$$

applies to the P pair.

Constraints and constraint Jacobian: The P pair entails $m_{\text{ext}}^{(P)} = 5$ external constraint functions that may be written in the form

$$\boldsymbol{\Phi}_{\text{ext}}^{(P)}(\mathbf{q}) = \begin{bmatrix} (\mathbf{m}_1^1)^{\mathrm{T}}(\mathbf{p}^2 - \mathbf{p}^1) \\ (\mathbf{m}_2^1)^{\mathrm{T}}(\mathbf{p}^2 - \mathbf{p}^1) \\ (\mathbf{d}_1^1)^{\mathrm{T}}\mathbf{d}_2^2 - \eta_1 \\ (\mathbf{d}_2^1)^{\mathrm{T}}\mathbf{d}_3^2 - \eta_2 \\ (\mathbf{d}_3^1)^{\mathrm{T}}\mathbf{d}_1^2 - \eta_2 \end{bmatrix} \tag{129}$$

where the η_i's are constant, and need to be consistent with the initial conditions. Again $\mathbf{m}_\alpha^1 \in \mathbb{R}^3$ and $\mathbf{p}^\alpha \in \mathbb{R}^3$ are given by (97) and (99), respectively. Note that the constraints resulting from the last three components of (129) conform with (128). The constraint Jacobian emanating from (129) is given by the 5×24 matrix

$$\mathbf{G}_{\text{ext}}^{(P)}(\mathbf{q}) = \begin{bmatrix} -(\mathbf{m}_1^1)^{\mathrm{T}} & \mathbf{G}_{11}^{\mathrm{T}} & \mathbf{G}_{12}^{\mathrm{T}} & \mathbf{G}_{13}^{\mathrm{T}} & (\mathbf{m}_1^1)^{\mathrm{T}} & \varrho_1^2(\mathbf{m}_1^1)^{\mathrm{T}} & \varrho_2^2(\mathbf{m}_1^1)^{\mathrm{T}} & \varrho_3^2(\mathbf{m}_1^1)^{\mathrm{T}} \\ -(\mathbf{m}_2^1)^{\mathrm{T}} & \mathbf{G}_{21}^{\mathrm{T}} & \mathbf{G}_{22}^{\mathrm{T}} & \mathbf{G}_{23}^{\mathrm{T}} & (\mathbf{m}_2^1)^{\mathrm{T}} & \varrho_1^2(\mathbf{m}_2^1)^{\mathrm{T}} & \varrho_2^2(\mathbf{m}_2^1)^{\mathrm{T}} & \varrho_3^2(\mathbf{m}_2^1)^{\mathrm{T}} \\ \mathbf{0}^{\mathrm{T}} & (\mathbf{d}_2^2)^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & (\mathbf{d}_1^1)^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & (\mathbf{d}_3^2)^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & (\mathbf{d}_2^1)^{\mathrm{T}} \\ \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & (\mathbf{d}_1^2)^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & (\mathbf{d}_3^1)^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} \end{bmatrix} \tag{130}$$

where the $\mathbf{G}_{\alpha i}$'s are again given by (102).

Discrete null space matrix: To get proper representations of both the continuous and the discrete null space matrices for the P pair, we adhere to the previous treatment of the C pair. To this end, one essentially has to remove $\dot{\theta}^2$, so that only $\dot{u}^2$ remains to characterize the motion of the second body relative to the first one ($r^{(P)} = 1$). Then (106) yields

$$\mathbf{v}_{\varphi}^2 = \mathbf{v}_{\varphi}^1 + \boldsymbol{\omega}^1 \times (\boldsymbol{\varphi}^2 - \boldsymbol{\varphi}^1) + \dot{u}^2 \mathbf{n}^1 \tag{131}$$

such that

$$\mathbf{P}_{\text{ext}}^{2,(P)}(\mathbf{q}) = \begin{bmatrix} \mathbf{I} & \widehat{\boldsymbol{\varphi}^1 - \boldsymbol{\varphi}^2} & \mathbf{n}^1 \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix} \tag{132}$$

Analogous to (113), the discrete version of (132) is given by

$$\mathbf{P}_{\text{ext}}^{2,(P)}(\mathbf{q}_n, \mathbf{q}_{n+1}) = \begin{bmatrix} \mathbf{I} & \widehat{\boldsymbol{\varphi}_{n+1/2}^1 - \boldsymbol{\varphi}_{n+1/2}^2} & (\mathbf{m}_1^1)_{n+1/2} \times (\mathbf{m}_2^1)_{n+1/2} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix} \tag{133}$$

such that the discrete null space matrix for the P pair can be written as

$$\mathbf{P}^{(P)}(\mathbf{q}_n, \mathbf{q}_{n+1}) = \begin{bmatrix} \mathbf{P}_{\text{int}}^1(\mathbf{q}_{n+1/2}^1) & \mathbf{0}_{6\times 1} \\ \multicolumn{2}{c}{\mathbf{P}_{\text{int}}^2(\mathbf{q}_{n+1/2}^2)\mathbf{P}_{\text{ext}}^{2,(P)}(\mathbf{q}_n, \mathbf{q}_{n+1})} \end{bmatrix} \tag{134}$$

where

$$\mathbf{P}_{\text{int}}^2(\mathbf{q}_{n+1/2}^2)\mathbf{P}_{\text{ext}}^{2,(P)}(\mathbf{q}_n, \mathbf{q}_{n+1}) = \begin{bmatrix} \mathbf{I} & \widehat{\boldsymbol{\varphi}_{n+1/2}^1 - \boldsymbol{\varphi}_{n+1/2}^2} & (\mathbf{m}_1^1)_{n+1/2} \times (\mathbf{m}_2^1)_{n+1/2} \\ \mathbf{0} & -(\widehat{\mathbf{d}_1^2})_{n+1/2} & \mathbf{0} \\ \mathbf{0} & -(\widehat{\mathbf{d}_2^2})_{n+1/2} & \mathbf{0} \\ \mathbf{0} & -(\widehat{\mathbf{d}_3^2})_{n+1/2} & \mathbf{0} \end{bmatrix} \tag{135}$$

Reparametrization of unknowns: The mapping $\mathbf{F}_{q_n}^{2,(P)}(\boldsymbol{\mu}^{(P)})$ can be inferred from the corresponding one for the C pair, Equation (118), by setting $\theta^2 = 0$. Accordingly,

$$\mathbf{q}_{n+1}^2 = \mathbf{F}_{q_n}^{2,(P)}(\boldsymbol{\mu}^{(P)}) = \begin{bmatrix} \boldsymbol{\varphi}_n^1 + \mathbf{u}_\varphi^1 + \exp(\widehat{\boldsymbol{\theta}^1})[\boldsymbol{\varrho}_n^1 - \boldsymbol{\varrho}_n^2 + (u_n^2 + u^2)\mathbf{n}_n^1] \\ \exp(\widehat{\boldsymbol{\theta}^1})(\mathbf{d}_1^2)_n \\ \exp(\widehat{\boldsymbol{\theta}^1})(\mathbf{d}_2^2)_n \\ \exp(\widehat{\boldsymbol{\theta}^1})(\mathbf{d}_3^2)_n \end{bmatrix} \tag{136}$$

with incremental unknowns $\boldsymbol{\mu}^{(P)} = (\mathbf{u}_\varphi^1, \boldsymbol{\theta}^1, u^2) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$.

4.5.5. Planar pair. As before in the context of the cylindrical pair, for the E pair (Figure 6), we make use of the orthonormal frame $\{\mathbf{m}_1^1, \mathbf{m}_2^1, \mathbf{n}^1\}$, with $\mathbf{n}^1 = n_i^1 \mathbf{d}_i^1$ and $\mathbf{m}_\alpha^1 = (m_\alpha^1)_i \mathbf{d}_i^1$. In the present case, the motion of the second body relative to the first one can be characterized by $r^{(E)} = 3$ DOF. Specifically, the second body may rotate about the axis specified by $\mathbf{n}^1$ and translate in the plane spanned by $\mathbf{m}_1^1$ and $\mathbf{m}_2^1$. Correspondingly, the rotational motion can be described by the kinematical relationship

$$\boldsymbol{\omega}^2 = \boldsymbol{\omega}^1 + \dot{\theta}^2 \mathbf{n}^1 \tag{137}$$

whereas, the relative translational motion may be accounted for by two coordinates $(u_1^2, u_2^2) \in \mathbb{R}^2$, such that

$$\mathbf{p}^2 = \mathbf{p}^1 + u_\alpha^2 \mathbf{m}_\alpha^1 \tag{138}$$

Note that, as before, $\mathbf{p}^\alpha = \boldsymbol{\varphi}^\alpha + \boldsymbol{\varrho}^\alpha$ for $\alpha = 1, 2$.

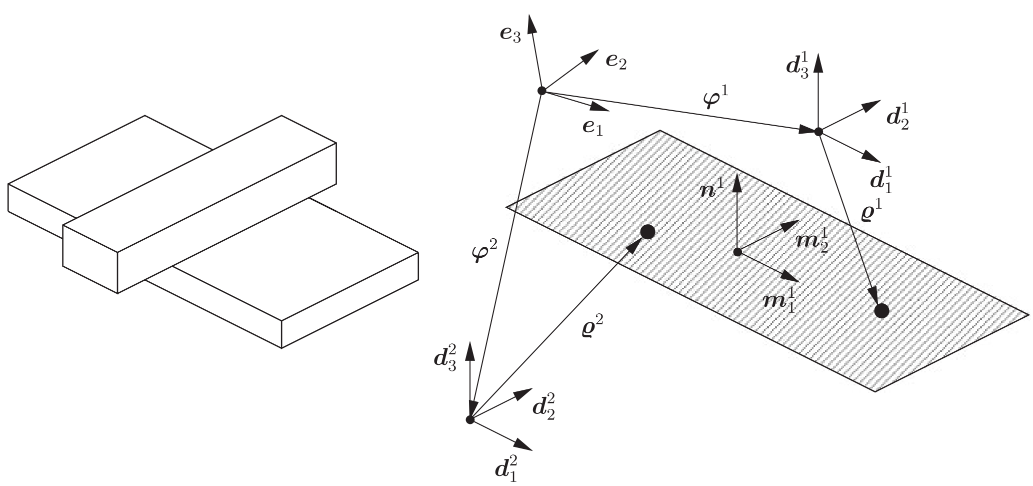

Figure 6. Planar pair.

Constraints and constraint Jacobian: The E pair gives rise to $m_{\text{ext}}^{(E)}=3$ external constraint functions, that may be written in the form

$$\mathbf{\Phi}_{\text{ext}}^{(E)}(\mathbf{q})=\begin{bmatrix}(\mathbf{n}^1)^{\text{T}}(\mathbf{p}^2-\mathbf{p}^1)\\ (\mathbf{n}^1)^{\text{T}}\mathbf{d}_1^2-\eta_1\\ (\mathbf{n}^1)^{\text{T}}\mathbf{d}_2^2-\eta_2\end{bmatrix} \tag{139}$$

where η_1,η_2 are constant, and need to be consistent with the initial conditions. Note that the first component of (139) conforms with (138), whereas the last two components of (139) conform with (137). The constraint Jacobian emanating from (139) is given by, the 3×24 matrix

$$\mathbf{G}_{\text{ext}}^{(E)}(\mathbf{q})=\begin{bmatrix}-(\mathbf{n}^1)^{\text{T}} & \mathbf{G}_1^{\text{T}} & \mathbf{G}_2^{\text{T}} & \mathbf{G}_3^{\text{T}} & (\mathbf{n}^1)^{\text{T}} & \varrho_1^2(\mathbf{n}^1)^{\text{T}} & \varrho_2^2(\mathbf{n}^1)^{\text{T}} & \varrho_3^2(\mathbf{n}^1)^{\text{T}}\\ \mathbf{0}^{\text{T}} & n_1^1(\mathbf{d}_1^2)^{\text{T}} & n_2^1(\mathbf{d}_1^2)^{\text{T}} & n_3^1(\mathbf{d}_1^2)^{\text{T}} & \mathbf{0}^{\text{T}} & (\mathbf{n}^1)^{\text{T}} & \mathbf{0}^{\text{T}} & \mathbf{0}^{\text{T}}\\ \mathbf{0}^{\text{T}} & n_1^1(\mathbf{d}_2^2)^{\text{T}} & n_2^1(\mathbf{d}_2^2)^{\text{T}} & n_3^1(\mathbf{d}_2^2)^{\text{T}} & \mathbf{0}^{\text{T}} & \mathbf{0}^{\text{T}} & (\mathbf{n}^1)^{\text{T}} & \mathbf{0}^{\text{T}}\end{bmatrix} \tag{140}$$

with

$$\mathbf{G}_i=n_i^1(\mathbf{p}^2-\mathbf{p}^1)-\varrho_i^1\mathbf{n}^1 \tag{141}$$

for $i=1,2,3$.

Continuous form of the null space matrix: Differentiating (138) with respect to time and taking into account (137), we obtain

$$\mathbf{v}_\varphi^2=\mathbf{v}_\varphi^1+\boldsymbol{\omega}^1\times(\boldsymbol{\varphi}^2-\boldsymbol{\varphi}^1)+\dot{u}_\alpha^2\mathbf{m}_\alpha^1+\dot{\theta}^2\boldsymbol{\varrho}^2\times\mathbf{n}^1 \tag{142}$$

The last equation in conjunction with (137) indicates that the twist of the second body can be expressed in terms of the independent velocities $\mathbf{v}^{(E)}=[\mathbf{t}^1,\dot{u}_1^2,\dot{u}_2^2,\dot{\theta}^2]^{\text{T}}$, such that $\mathbf{t}^{2,(E)}=\mathbf{P}_{\text{ext}}^{2,(E)}\mathbf{v}^{(E)}$. Here, the 6×9 matrix $\mathbf{P}_{\text{ext}}^{2,(E)}$ is given by

$$\mathbf{P}_{\text{ext}}^{2,(E)}(\mathbf{q})=\begin{bmatrix}\mathbf{I} & \widehat{\boldsymbol{\varphi}^1-\boldsymbol{\varphi}^2} & \mathbf{m}_1^1 & \mathbf{m}_2^1 & \boldsymbol{\varrho}^2\times\mathbf{n}^1\\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{n}^1\end{bmatrix} \tag{143}$$

Then (59) yields

$$\mathbf{P}_{\text{ext}}^{(E)}=\begin{bmatrix}\mathbf{I}_{6\times 6} & \mathbf{0}_{6\times 3}\\ \multicolumn{2}{c}{\mathbf{P}_{\text{ext}}^{2,(E)}}\end{bmatrix} \tag{144}$$

Finally, with regard to (61), the null space matrix for the E pair is given by

$$\mathbf{P}^{(E)}(\mathbf{q})=\begin{bmatrix}\mathbf{P}_{\text{int}}^1 & \mathbf{0}_{6\times 2}\\ \multicolumn{2}{c}{\mathbf{P}_{\text{int}}^2\mathbf{P}_{\text{ext}}^{2,(E)}}\end{bmatrix} \tag{145}$$

with

$$\mathbf{P}^2_{\text{int}}\mathbf{P}^{2,(E)}_{\text{ext}} = \begin{bmatrix} \mathbf{I} & \widehat{\boldsymbol{\varphi}^1 - \boldsymbol{\varphi}^2} & \mathbf{m}^1_1 & \mathbf{m}^1_2 & \boldsymbol{\varrho}^2 \times \mathbf{n}^1 \\ \mathbf{0} & -\widehat{\mathbf{d}^2_1} & \mathbf{0} & \mathbf{0} & \mathbf{n}^1 \times \mathbf{d}^2_1 \\ \mathbf{0} & -\widehat{\mathbf{d}^2_2} & \mathbf{0} & \mathbf{0} & \mathbf{n}^1 \times \mathbf{d}^2_2 \\ \mathbf{0} & -\widehat{\mathbf{d}^2_3} & \mathbf{0} & \mathbf{0} & \mathbf{n}^1 \times \mathbf{d}^2_3 \end{bmatrix} \tag{146}$$

For later use, we calculate matrix (64), which in the present case is given by the 3×12 matrix

$$\widetilde{\mathbf{G}}^{(E)}_{\text{ext}} = \mathbf{G}^{(E)}_{\text{ext}}\mathbf{P}_{\text{int}} = \begin{bmatrix} -(\mathbf{n}^1)^{\text{T}} & -\mathbf{G}^{\text{T}}_i\widehat{\mathbf{d}^1_i} & (\mathbf{n}^1)^{\text{T}} & -(\mathbf{n}^1)^{\text{T}}\widehat{\boldsymbol{\varrho}^2} \\ \mathbf{0}^{\text{T}} & -(\mathbf{d}^2_1)^{\text{T}}\widehat{\mathbf{n}^1} & \mathbf{0}^{\text{T}} & -(\mathbf{n}^1)^{\text{T}}\widehat{\mathbf{d}^2_1} \\ \mathbf{0}^{\text{T}} & -(\mathbf{d}^2_2)^{\text{T}}\widehat{\mathbf{n}^1} & \mathbf{0}^{\text{T}} & -(\mathbf{n}^1)^{\text{T}}\widehat{\mathbf{d}^2_2} \end{bmatrix} \tag{147}$$

It can be easily checked by a straightforward calculation that the present design procedure for $\mathbf{P}^{2,(E)}_{\text{ext}}$ ensures that

$$\widetilde{\mathbf{G}}^{(E)}_{\text{ext}}\mathbf{P}^{(E)}_{\text{ext}} = \mathbf{0} \tag{148}$$

Discrete version of the null space matrix: In the present case, the discrete null space matrix does not coincide with $\mathbf{P}^{(E)}(\mathbf{q}_{n+1/2})$. Instead, applying the procedure described in Section 4.3, we choose

$$\mathbf{P}^{2,(E)}_{\text{ext}}(\mathbf{q}_n, \mathbf{q}_{n+1}) = \begin{bmatrix} \mathbf{I} & (\widehat{\boldsymbol{\varphi}^1 - \boldsymbol{\varphi}^2})_{n+1/2} & (\mathbf{m}^1_2)_{n+1/2} \times (\mathbf{n}^1)_{n+1/2} & (\mathbf{n}^1)_{n+1/2} \times (\mathbf{m}^1_1)_{n+1/2} & \boldsymbol{\varrho}^2_{n+1/2} \times \mathbf{n}^1_{n+1/2} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{n}^1_{n+1/2} \end{bmatrix} \tag{149}$$

It can be easily verified that (149) satisfies the design conditions (74). Finally, in view of (70), the discrete null space matrix for the E pair assumes the form

$$\mathbf{P}^{(E)}(\mathbf{q}_n, \mathbf{q}_{n+1}) = \begin{bmatrix} \mathbf{P}^1_{\text{int}}(\mathbf{q}^1_{n+1/2}) & \mathbf{0}_{6\times 3} \\ \mathbf{P}^2_{\text{int}}(\mathbf{q}^2_{n+1/2})\mathbf{P}^{2,(E)}_{\text{ext}}(\mathbf{q}_n, \mathbf{q}_{n+1}) & \end{bmatrix} \tag{150}$$

where

$$
\mathbf{P}^2_{\text{int}}(\mathbf{q}^2_{n+1/2})\mathbf{P}^{2,(E)}_{\text{ext}}(\mathbf{q}_n, \mathbf{q}_{n+1})
$$

$$
= \begin{bmatrix} \mathbf{I} & (\widehat{\boldsymbol{\varphi}^1 - \boldsymbol{\varphi}^2})_{n+1/2} & (\mathbf{m}^1_2)_{n+1/2} \times (\mathbf{n}^1)_{n+1/2} & (\mathbf{n}^1)_{n+1/2} \times (\mathbf{m}^1_1)_{n+1/2} & \boldsymbol{\varrho}^2_{n+1/2} \times \mathbf{n}^1_{n+1/2} \\ \mathbf{0} & -(\widehat{\mathbf{d}^2_1})_{n+1/2} & \mathbf{0} & \mathbf{0} & \mathbf{n}^1_{n+1/2} \times (\mathbf{d}^2_1)_{n+1/2} \\ \mathbf{0} & -(\widehat{\mathbf{d}^2_2})_{n+1/2} & \mathbf{0} & \mathbf{0} & \mathbf{n}^1_{n+1/2} \times (\mathbf{d}^2_2)_{n+1/2} \\ \mathbf{0} & -(\widehat{\mathbf{d}^2_3})_{n+1/2} & \mathbf{0} & \mathbf{0} & \mathbf{n}^1_{n+1/2} \times (\mathbf{d}^2_3)_{n+1/2} \end{bmatrix} \tag{151}
$$

Reparametrization of unknowns: In the present case, the configuration of the second body with respect to the first one can be characterized by the incremental variables $\boldsymbol{\vartheta}^{(E)} = (u^2_1, u^2_2, \theta^2) \in \mathbb{R}^3$. Here, θ^2 accounts for the incremental relative rotation which may be expressed via the product of exponentials formula

$$
(\mathbf{d}^2_I)_{n+1} = \exp(\widehat{\boldsymbol{\theta}^1}) \exp(\theta^2 (\widehat{\mathbf{n}^1})_n)(\mathbf{d}^2_I)_n \tag{152}
$$

Enforcing the external constraints (138) at the end of the time step implies

$$
\boldsymbol{\varphi}^2_{n+1} = \boldsymbol{\varphi}^1_{n+1} + \boldsymbol{\varrho}^1_{n+1} - \boldsymbol{\varrho}^2_{n+1} + ((u^2_\alpha)_n + u^2_\alpha)(\mathbf{m}^1_\alpha)_{n+1} \tag{153}
$$

Accordingly, the mapping $\mathbf{F}^{2,(E)}_{q_n}(\boldsymbol{\mu}^{(E)})$ can be written in the form

$$
\mathbf{q}^2_{n+1} = \mathbf{F}^{2,(E)}_{q_n}(\boldsymbol{\mu}^{(E)}) = \begin{bmatrix} \boldsymbol{\varphi}^1_n + \mathbf{u}^1_\varphi + \exp(\widehat{\boldsymbol{\theta}^1})[\boldsymbol{\varrho}^1_n - \exp(\theta^2(\widehat{\mathbf{n}^1})_n)\boldsymbol{\varrho}^2_n + ((u^2_\alpha)_n + u^2_\alpha)(\mathbf{m}^1_\alpha)_n] \\ \exp(\widehat{\boldsymbol{\theta}^1}) \exp(\theta^2(\widehat{\mathbf{n}^1})_n)(\mathbf{d}^2_1)_n \\ \exp(\widehat{\boldsymbol{\theta}^1}) \exp(\theta^2(\widehat{\mathbf{n}^1})_n)(\mathbf{d}^2_2)_n \\ \exp(\widehat{\boldsymbol{\theta}^1}) \exp(\theta^2(\widehat{\mathbf{n}^1})_n)(\mathbf{d}^2_3)_n \end{bmatrix} \tag{154}
$$

5. NUMERICAL EXAMPLES

5.1. Steady precession of a gyro top

We consider the motion of a symmetrical top with a fixed point on its axis of symmetry (Figure 7). The top can be modelled as spherical pair, where the first body is fixed in space and the second body coincides with the top. Accordingly, with regard to Section 4.5.1, the configuration of the top is characterized by $\mathbf{q}^2 \in \mathbb{R}^{12}$. In the following, we omit the superscript 2 such that, for example, the relevant coordinates of the top are given by

$$
\mathbf{q} = (\boldsymbol{\varphi}, \{\mathbf{d}_I\}) \in \mathbb{R}^{12} \tag{155}
$$

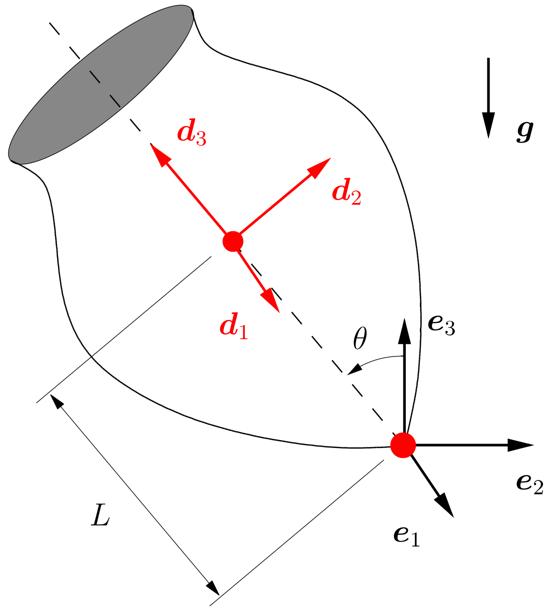

Figure 7. Symmetrical top.

The shape of the top is assumed to be a cone with height $H = 0.1$ and radius $R = 0.05$. The centre of mass is located at $L = \frac{3}{4}H$, so that the location of the spherical joint with respect to the body frame is given by

$$\boldsymbol{\varrho} = \varrho_i \mathbf{d}_i \quad \text{with} \quad [\varrho_i] = [0, 0, -L] \tag{156}$$

The total mass of the top is $M_\varphi = \frac{1}{3}\varrho \pi R^2 H$, the principal inertias with respect to the centre of mass are

$$J_1 = J_2 = \frac{3M_\varphi}{80}(4R^2 + H^2) \quad \text{and} \quad J_3 = \frac{3M_\varphi}{10}R^2 \tag{157}$$

and the mass density is assumed to be $\varrho = 2700$. Then the principal values of the Euler tensor with respect to the centre of mass follow from

$$\mathbf{E} = \frac{1}{2}(\operatorname{tr}\mathbf{J})\mathbf{I} - \mathbf{J} \tag{158}$$

such that, the mass matrix $\mathbf{M} \in \mathbb{R}^{12\times 12}$ in (23) can be easily set up. Gravity is acting on the top, such that the potential energy function is given by

$$V(\mathbf{q}) = M_\varphi g \boldsymbol{\varphi}^{\mathrm{T}} \mathbf{e}_3 \quad \text{with} \quad g = 9.81 \tag{159}$$

5.1.1. Constrained scheme. In addition to the $n = 12$ redundant coordinates, the constrained scheme relies on $m = 9$ multipliers associated with $m_{\text{int}} = 6$ internal constraints along with $m_{\text{ext}} = 3$ external constraints. While the internal constraints are given by (24), the external constraints can be easily extracted from the treatment of the spherical pair in Section 4.5.1. Accordingly, the external constraints assume the form

$$\mathbf{\Phi}_{\text{ext}}(\mathbf{q}) = \boldsymbol{\varphi} + \boldsymbol{\varrho} = \mathbf{0} \tag{160}$$

with corresponding 3×12 constraint Jacobian

$$\mathbf{G}_{\text{ext}}(\mathbf{q}) = [\mathbf{I} \quad \varrho_1 \mathbf{I} \quad \varrho_2 \mathbf{I} \quad \varrho_3 \mathbf{I}] \tag{161}$$

Obviously, the number of unknowns of the constrained scheme amounts to $n + m = 21$.

5.1.2. Reduced scheme. Application of the discrete null space method leads to the reduced scheme with $n - m = 3$ unknowns. These unknowns correspond to the incremental rotation vector $\boldsymbol{\theta} \in \mathbb{R}^3$. In particular, in the present case, reparametrization (95) of the spherical pair yields

$$\mathbf{q}_{n+1} = \mathbf{F}_{q_n}(\boldsymbol{\theta}) = \begin{bmatrix} -\exp(\widehat{\boldsymbol{\theta}})\boldsymbol{\varrho}_n \\ \exp(\widehat{\boldsymbol{\theta}})(\mathbf{d}_1)_n \\ \exp(\widehat{\boldsymbol{\theta}})(\mathbf{d}_2)_n \\ \exp(\widehat{\boldsymbol{\theta}})(\mathbf{d}_3)_n \end{bmatrix} \tag{162}$$

The treatment of the spherical pair indicates that, in the present case, the independent generalized velocities coincide with the angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ of the top. Furthermore, the present null space matrix can be extracted from that of the spherical pair, Equation (86). Accordingly,

$$\mathbf{P}(\mathbf{q}) = \begin{bmatrix} \widehat{\boldsymbol{\varrho}} \\ -\widehat{\mathbf{d}_1} \\ -\widehat{\mathbf{d}_2} \\ -\widehat{\mathbf{d}_3} \end{bmatrix} \tag{163}$$

It further follows from the treatment of the S pair, that the discrete null space matrix for the gyro top is given by (163), evaluated in the mid-point configuration $\mathbf{q}_{n+1/2}$.

5.1.3. Initial conditions. In order to provide an analytical reference solution, we consider the case of precession with no nutation. Let θ be the angle of nutation, ω_{p} the precession rate and ω_{s} the spin rate. Specifically, as initial values we choose

$$\theta = \frac{\pi}{3} \quad \text{and} \quad \omega_{\text{p}} = 10 \tag{164}$$

The condition for steady precession can be written as (see, for example, Reference [26, Section 5.3])

$$\omega_{\text{s}} = \frac{M_{\varphi} g L}{\tilde{M}_3 \omega_{\text{p}}} + \frac{\tilde{M}_1 - \tilde{M}_3}{\tilde{M}_3} \omega_{\text{p}} \cos\theta \tag{165}$$

Here, $\tilde{M}_I$ are the principal values of the reduced mass matrix (8). In the present case, we get

$$\widetilde{\mathbf{M}} = \mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{P} = \mathbf{J} + M_{\varphi}(\|\boldsymbol{\varrho}\|^2\mathbf{I} - \boldsymbol{\varrho}\otimes\boldsymbol{\varrho}) \tag{166}$$

or, in view of (156),

$$\widetilde{\mathbf{M}} = \sum_{I=1}^{3} J_I \mathbf{d}_I \otimes \mathbf{d}_I + M_{\varphi} L^2(\mathbf{d}_1 \otimes \mathbf{d}_1 + \mathbf{d}_2 \otimes \mathbf{d}_2) \tag{167}$$

Accordingly, the principal values of the reduced mass matrix to be inserted into (165) are given by

$$\tilde{M}_1 = J_1 + M_{\varphi} L^2 \quad \text{and} \quad \tilde{M}_3 = J_3 \tag{168}$$

Note that the reduced mass matrix conforms to the well-known parallel-axis theorem. With regard to the initial values (164) and (165), consistent initial values for the redundant coordinates and the corresponding velocities can be calculated next. Accordingly, the initial configuration is characterized by $\mathbf{q} \in \mathbb{R}^{12}$ with

$$\mathbf{d}_I = \exp(\theta\widehat{\mathbf{e}_1})\mathbf{e}_I \quad \text{and} \quad \boldsymbol{\varphi} = -\boldsymbol{\varrho} = L\mathbf{d}_3 \tag{169}$$

Consistent initial velocities $\mathbf{v} \in \mathbb{R}^{12}$ can be calculated by employing the null space matrix (163). Accordingly,

$$\mathbf{v} = \mathbf{P}(\mathbf{q})\boldsymbol{\omega} \tag{170}$$

with initial angular velocity vector given by

$$\boldsymbol{\omega} = \omega_{\mathrm{p}}\mathbf{e}_3 + \omega_{\mathrm{s}}\mathbf{d}_3 \tag{171}$$

To summarize, the analytical reference solution for the present motion of the gyro top corresponds to steady precession with constant angle of nutation $\theta = \pi/3$. As further reference values, we compute the total energy E as well as the component $\mathbf{e}_3^{\mathrm{T}}\mathbf{L}$ of the angular momentum vector $\mathbf{L}$. In the present case, both quantities are first integrals of the motion. Making use of the kinetic energy in form (7) along with the potential energy function (159), the total energy can be written as

$$E(\mathbf{q}, \boldsymbol{\omega}) = \tfrac{1}{2}\boldsymbol{\omega}^{\mathrm{T}}\widetilde{\mathbf{M}}\boldsymbol{\omega} + V(\mathbf{q}) \tag{172}$$

In addition to that, the angular momentum vector (with respect to the origin of the reference frame $\{\mathbf{e}_I\}$) is given by

$$\mathbf{L} = \widetilde{\mathbf{M}}\boldsymbol{\omega} \tag{173}$$

5.1.4. Numerical results

Convergence properties: To visualize the convergence properties of the present algorithm, we compare the numerical results with the analytical reference solution. To this end, we consider the motion of the centre of mass. In particular, Figures 8 and 9, depict the x, z coordinates ($\boldsymbol{\varphi}(t) = x(t)\mathbf{e}_1 + y(t)\mathbf{e}_2 + z(t)\mathbf{e}_3$) of the mass centre versus time for $\Delta t = 0.05$ and 0.01, respectively. Convergence towards the analytical solution can be easily observed. For

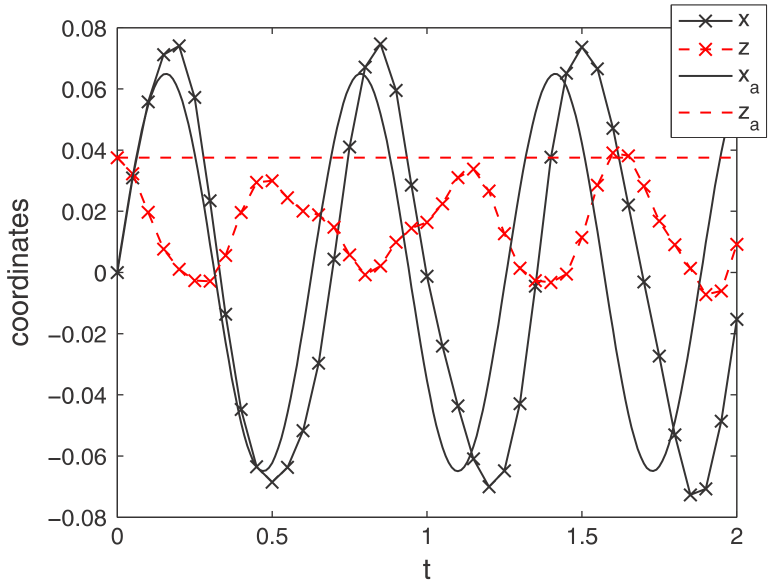

Figure 8. Steady precession of a gyro top: comparison of analytical solution $(x_a(t), z_a(t))$ and numerical solution $(x, z, \Delta t = 0.05)$ for the motion of the centre of mass.

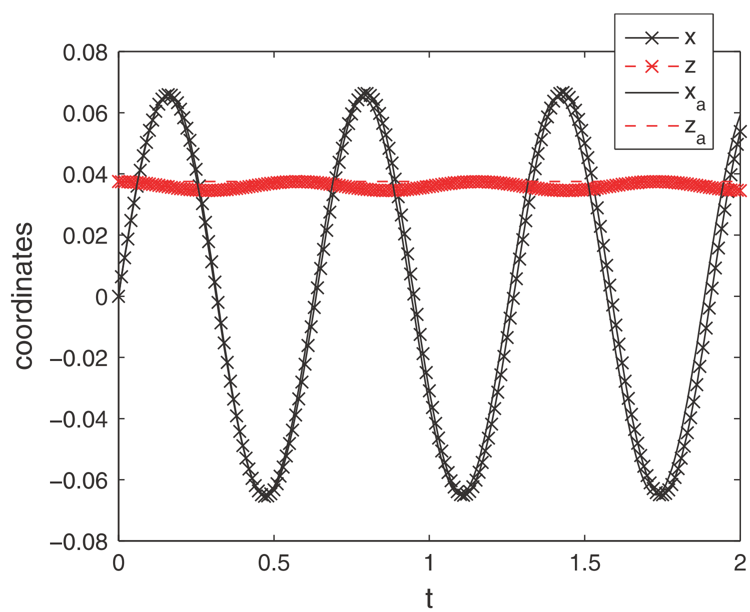

Figure 9. Steady precession of a gyro top: comparison of analytical solution $(x_a(t), z_a(t))$ and numerical solution $(x, z, \Delta t = 0.01)$ for the motion of the centre of mass.

Table II. Comparison of the condition number of the iteration matrix for the example 'steady precession of a gyro top'.

Δt	Constrained scheme	Reduced scheme
5×10^{-2}	$\approx 1.6\times 10^{4}$	≈ 8
5×10^{-3}	$\approx 1.6\times 10^{7}$	≈ 8
5×10^{-4}	$\approx 1.6\times 10^{10}$	≈ 8

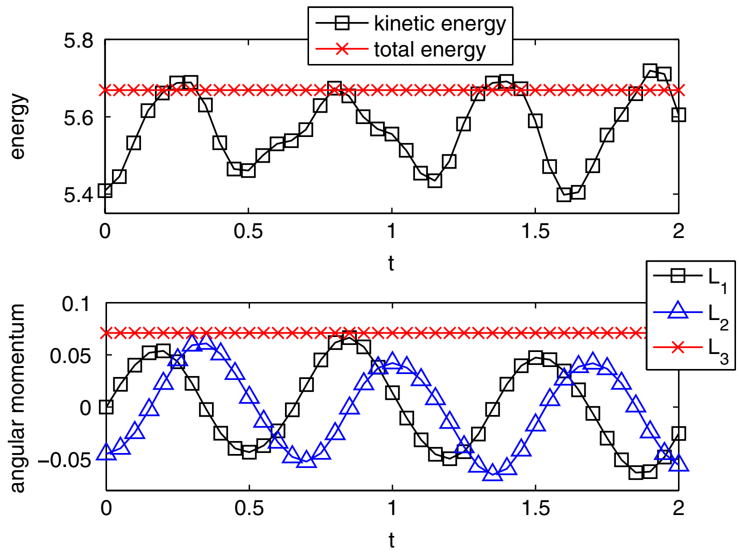

Figure 10. Steady precession of a gyro top: energy and components of angular momentum vector ($\Delta t = 0.05$).

$\Delta t = 0.001$, practically no deviation of the numerical results from the analytical solution can be observed anymore.

Conditioning: The condition number of the iteration matrix is depicted in Table II for both the constrained scheme and the reduced scheme. Accordingly, the condition number of the constrained scheme deteriorates with smaller time steps, whereas the reduced scheme remains well-conditioned.

Conservation properties: As depicted in Figure 10, the first integrals of the motion are indeed preserved by the present algorithm. That is, both the total energy and the 3-component of the angular momentum vector are conserved quantities.

5.2. Cylindrical pair

We next investigate the free flight of a cylindrical pair (Figure 11, see also Figure 3). The first body consists of a cylinder of length $l^1 = 30$, radius $r^1 = 2$ and mass $M^1_\varphi = 4$. The second body is modelled as a hollow cylinder of length $l^2 = 6$, outer radius $r^2_o = 3$, inner radius $r^2_i = 2$ and mass $M^2_\varphi = 3$. The principal values of the inertia tensor with respect to the centre of mass

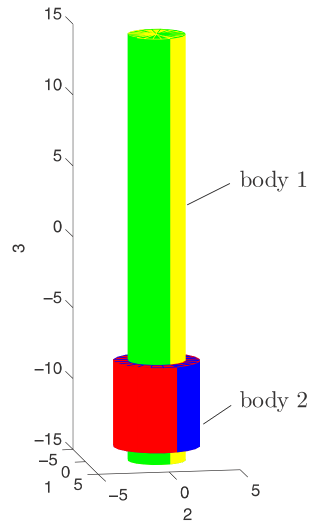

Figure 11. Initial configuration of the cylindrical pair.

are given by

$$[J_i^1] = [304, 304, 8]$$

and

$$[J_i^2] = [18.75, 18.75, 19.5]$$

respectively. The cylindrical joints are located in the centre of mass of each body. Consequently, relative to the respective body frame, the location is characterized by

$$[\varrho_i^1] = [0, 0, 0] \quad \text{and} \quad [\varrho_i^2] = [0, 0, 0]$$

Furthermore, the unit vector (96) is specified by

$$[n_i^1] = [0, 0, 1]$$

The initial configuration of the cylindrical pair is characterized by $\boldsymbol{\varphi}^\alpha = \varphi_i^\alpha \mathbf{e}_i$ with

$$[\varphi_i^1] = [0, 0, 0] \quad \text{and} \quad [\varphi_i^2] = [0, 0, -11]$$

along with

$$\mathbf{d}_i^1 = \mathbf{e}_i \quad \text{and} \quad \mathbf{d}_i^2 = \mathbf{e}_i$$

Note that corresponding consistent initial relative coordinates are $u^2 = -11$ and $\theta^2 = 0$. Consistent initial velocities can be computed by using the null space matrix (109), such that

$$\mathbf{v} = \mathbf{P}^{(C)} \mathbf{v}^{(C)} \tag{174}$$

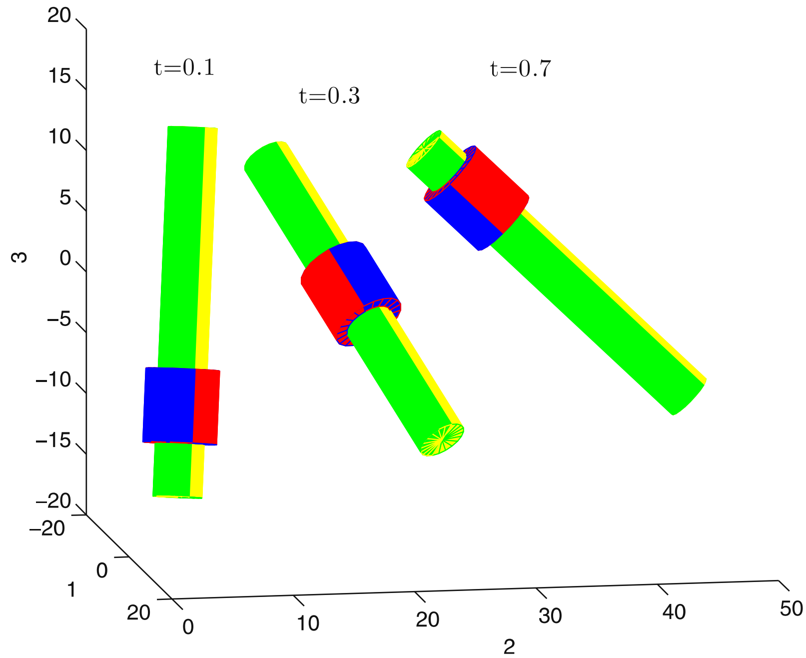

Figure 12. Cylindrical pair: snapshots of the motion at $t = 0.1, 0.3, 0.7$.

where the independent generalized velocities are specified by

$$\mathbf{v}^{(C)} = \begin{bmatrix} \mathbf{v}_{\varphi}^{1} \\ \boldsymbol{\omega}^{1} \\ \dot{u}^{2} \\ \dot{\theta}^{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 50 \\ 0 \\ 1 \\ 1.5 \\ 0 \\ 35.5 \\ -100 \end{bmatrix} \tag{175}$$

No external forces are acting on the C pair such that the total energy and the vector of angular momentum are first integrals of the motion. We further, remark that the implementation of the constrained scheme leads to $n + m^{(C)} = 24 + 16 = 40$ unknowns, whereas the application of the discrete null space method, yields a reduction to $n - m^{(C)} = 8$ unknowns (cf. Section 4).

To illustrate the motion of the C pair, Figure 12 shows some snapshots at $t \in \{0.1, 0.3, 0.7\}$. Figure 13 confirms algorithmic conservation of the total energy, and the components L_i of the

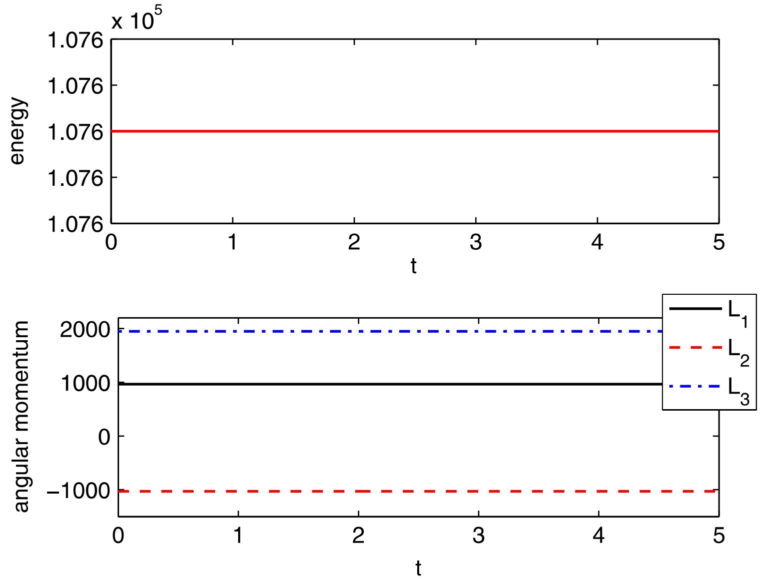

Figure 13. Cylindrical pair: energy and components of angular momentum vector $\mathbf{L} = L_i \mathbf{e}_i$ ($\Delta t = 0.01$).

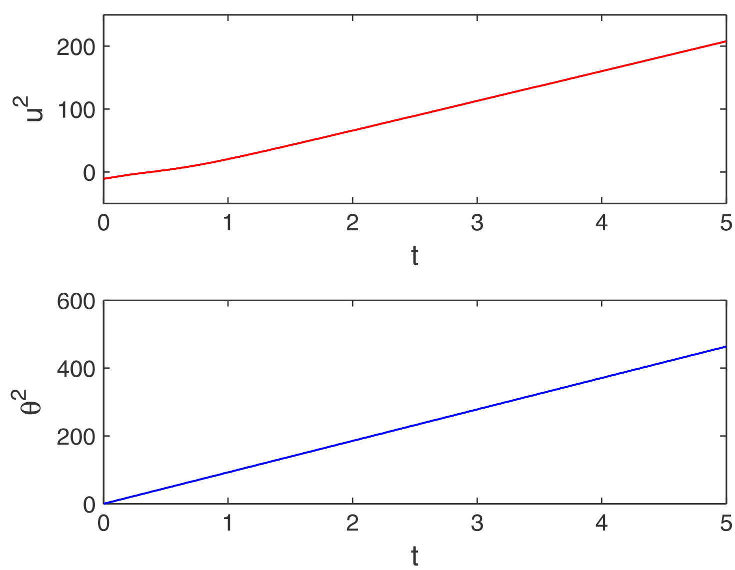

Figure 14. Cylindrical pair: relative coordinates $u^2(t)$ and $\theta^2(t)$ ($\Delta t = 0.01$).

angular momentum vector $\mathbf{L} = L_i \mathbf{e}_i$. Furthermore, the evolution of the relative DOF $u^2(t)$ and $\theta^2(t)$ is depicted in Figure 14.

From Table III, the condition number of the iteration matrix for the constrained scheme and the reduced scheme can be compared. Accordingly, the reduced scheme is well conditioned for

Table III. Comparison of the condition number of the iteration matrix for the example 'cylindrical pair'.

Δt	Constrained scheme	Reduced scheme
1×10^{-2}	$\approx 3.6\times 10^{11}$	≈ 360
1×10^{-3}	$\approx 3.6\times 10^{14}$	≈ 358
1×10^{-4}	$\approx 3.6\times 10^{17}$	≈ 358

all time steps, whereas the condition number of the constrained scheme increases heavily for decreasing time steps.

5.3. Planar pair

We next investigate the free flight of a planar pair (Figure 15, see also Figure 6). The E pair consists of a parallelepiped of mass $M^1_\varphi = 5$ and side lengths $l^1_x = l^1_y = 16$, $l^1_z = 0.5$, such that the principal values of the inertia tensor, with respect to the centre of mass, are given by

$$[J^1_i] = \left[\frac{5125}{48}, \frac{5125}{48}, \frac{640}{3}\right]$$

The second body is modelled as a pyramid with mass $M^2_\varphi = 2$, side length of the square base $l^2_x = l^2_y = 2$ and height $l^2_z = 3$, leading to the principal values of the inertia tensor with respect to the centre of mass

$$[J^2_i] = \left[\frac{43}{40}, \frac{43}{40}, \frac{4}{5}\right]$$

The location of the planar joint relative to the respective body frame is characterized by

$$[\varrho^1_i] = [0, 0, 0.25] \quad \text{and} \quad [\varrho^2_i] = [0, 0, -1]$$

Furthermore, the orthonormal frame needed for the description of the relative motion is specified by

$$[n^1_i] = [0, 0, 1], \quad [(m^1_1)_i] = [1, 0, 0], \quad [(m^1_2)_i] = [0, 1, 0]$$

such that the pyramid is constrained to slide on the top surface of the parallelepiped. The initial configuration of the planar pair is characterized by $\boldsymbol{\varphi}^\alpha = \varphi^\alpha_i \mathbf{e}_i$ with,

$$[\varphi^1_1] = [5, 5, 5] \quad \text{and} \quad [\varphi^2_i] = [-2, -2, 6.25]$$

along with

$$\mathbf{d}^1_i = \mathbf{e}_i \quad \text{and} \quad \mathbf{d}^2_i = \mathbf{e}_i$$

Correspondingly, consistent initial values for the relative coordinates are given by

$$u^2_1 = -7, \quad u^2_2 = -7, \quad \theta^2 = 0$$

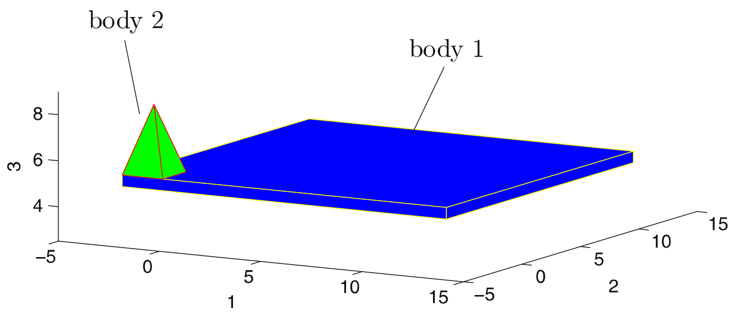

Figure 15. Initial configuration of the planar pair.

Consistent initial velocities are specified by using the null space matrix (145), so that

$$\mathbf{v} = \mathbf{P}^{(E)}\mathbf{v}^{(E)} \tag{176}$$

with independent generalized velocities

$$\mathbf{v}^{(E)} = \begin{bmatrix} \mathbf{v}_{\varphi}^{1} \\ \boldsymbol{\omega}^{1} \\ \dot{u}_1^2 \\ \dot{u}_2^2 \\ \dot{\theta}^2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ -20 \\ -20 \\ 10 \\ 150 \\ -120 \\ 60 \end{bmatrix} \tag{177}$$

Note that, for clearness of exposition, the initial velocity of the mass centre of the first body has been set to zero ($\mathbf{v}_{\varphi}^{1} = \mathbf{0}$). Snapshots of the planar pair at consecutive instants illustrate the simulated motion in Figure 16. Since no external forces are acting on the planar pair, both the energy and the vector of angular momentum are conserved quantities. The corresponding algorithmic conservation properties are confirmed in Figure 17. Furthermore, Figure 18 depicts the evolution of the relative coordinates specifying the configuration of the second body relative to the first one.

Table IV again verifies that the condition number of the iteration matrix, can be significantly improved by applying the discrete null space method. In this connection, we recall that the implementation of the constrained scheme is based on $n + m^{(E)} = 24 + 15 = 39$ unknowns, whereas the application of the discrete null space method yields a reduction to $n - m^{(E)} = 9$ unknowns (cf. Section 4).

5.4. *Kinematic chains*

The previous treatment of kinematic pairs can be directly extended to kinematic chains. The extension to kinematic chains is outlined next for both open and closed loop systems. Eventually, we present a numerical example dealing with a six-body linkage.

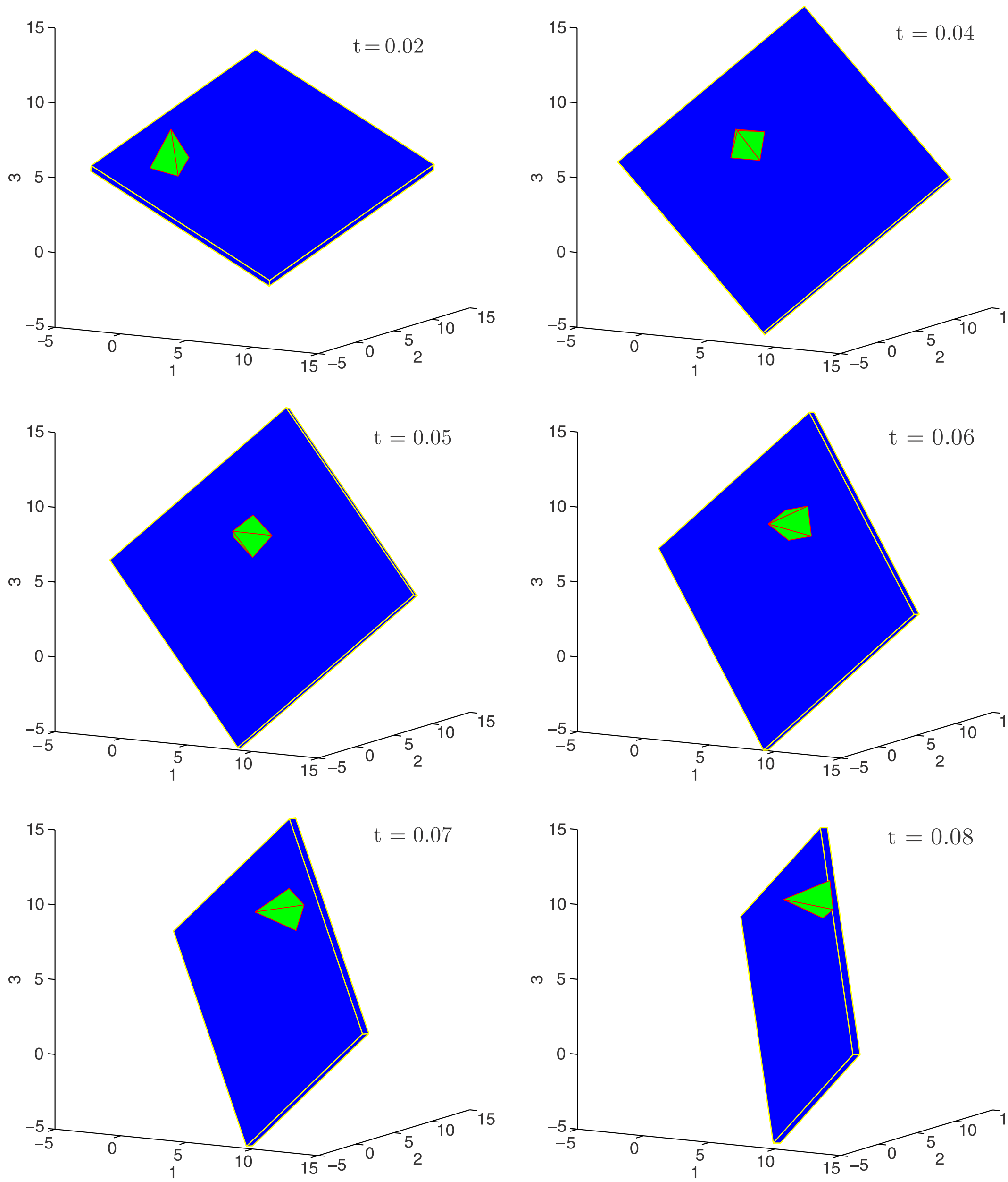

Figure 16. Planar pair: snapshots of the motion at $t \in \{0.02, 0.04, 0.05, 0.06, 0.07, 0.08\}$.

5.4.1. Open kinematic chain. In preparation for the numerical example, we first outline the treatment of a serial manipulator consisting of six rigid bodies interconnected by revolute joints (Figure 19). Body 0 is fixed in space such that, the present rigid body formulation

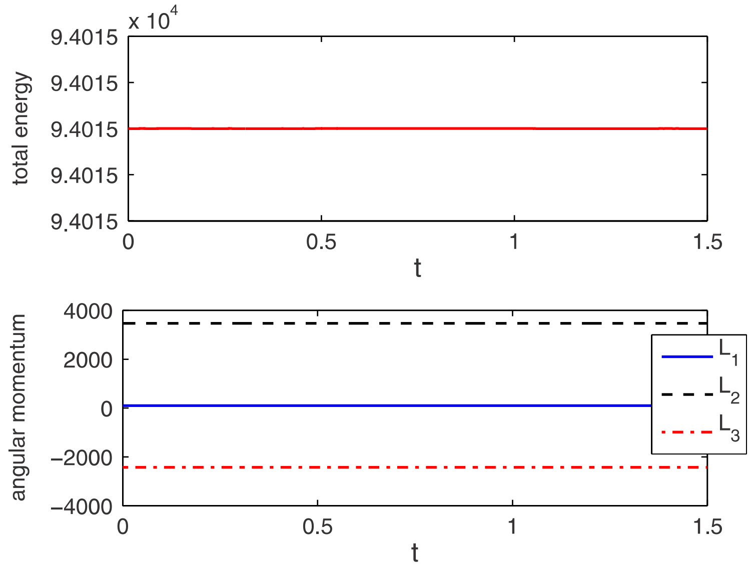

Figure 17. Planar pair: energy and components of angular momentum vector $\mathbf{L}=L_i\mathbf{e}_i$ $(\Delta t=0.01)$.

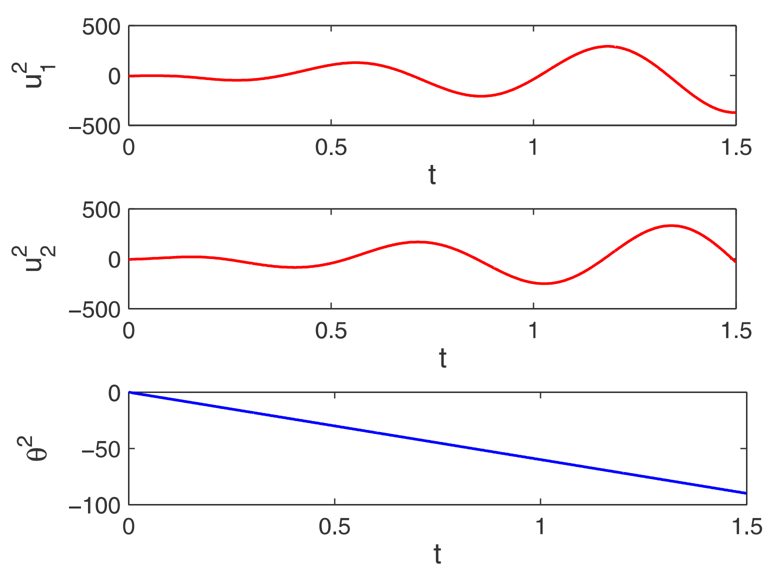

Figure 18. Planar pair: relative coordinates $u_1^2(t)$, $u_2^2(t)$ and $\theta^2(t)$ $(\Delta t=0.001)$.

implies $n=5\times 12=60$ redundant coordinates along with $m=m_{\text{int}}+m_{\text{ext}}$ constraints. Each rigid link is based on six internal constraints of form (24). Furthermore, each revolute joint yields five external constraints of form (119). Accordingly, in total there are $m=5\times 6+5\times 5=55$

Table IV. Comparison of the condition number of the iteration matrix for the example 'planar pair'.

Δt	Constrained scheme	Reduced scheme
1×10^{-2}	$\approx 4.8 \times 10^{10}$	$\approx 7.3 \times 10^{2}$
1×10^{-3}	$\approx 4.6 \times 10^{13}$	$\approx 8.0 \times 10^{2}$
1×10^{-4}	$\approx 4.6 \times 10^{16}$	$\approx 8.0 \times 10^{2}$

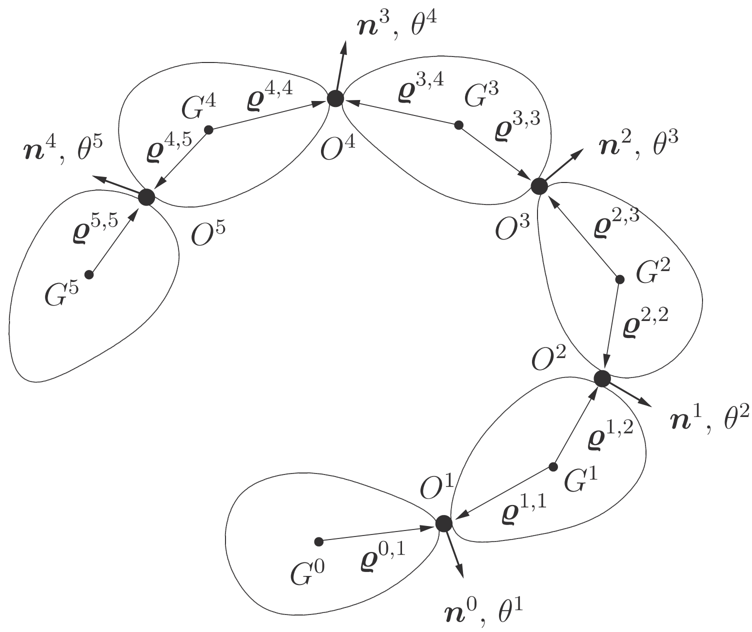

Figure 19. Schematic illustration of the open kinematic chain.

constraints associated with the kinematic chain under consideration. This corresponds to $n - m = 5$ DOF.

Null space matrix: The treatment of the revolute pair in Section 4.5.3 can now be easily generalized to the kinematic chain at hand. In view of (122), the relationship between the twist of the ith body and the $(i-1)$st body (Figure 20) can be written as

$$\mathbf{t}^i = \mathbf{P}_{\text{ext}}^{i,(R)} \begin{bmatrix} \mathbf{t}^{i-1} \\ \dot{\theta}^i \end{bmatrix} \tag{178}$$

Partitioning $\mathbf{P}_{\text{ext}}^{i,(R)} = [\mathbf{P}_{\text{ext}}^{i,a}, \mathbf{P}_{\text{ext}}^{i,b}]$, with $\mathbf{P}_{\text{ext}}^{i,a} \in \mathbb{R}^{6\times 6}$ and $\mathbf{P}_{\text{ext}}^{i,b} \in \mathbb{R}^{6\times 1}$, Equation (178) may be rewritten as

$$\mathbf{t}^i = \mathbf{P}_{\text{ext}}^{i,a}\mathbf{t}^{i-1} + \mathbf{P}_{\text{ext}}^{i,b}\dot{\theta}^i \tag{179}$$

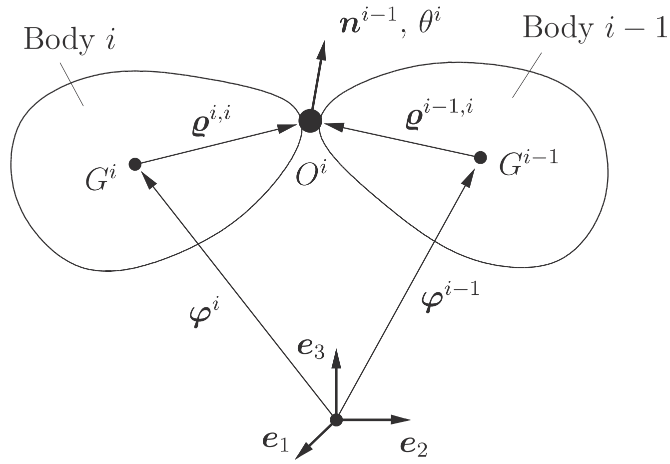

Figure 20. Revolute pair in a simple kinematic chain.

where

$$\mathbf{P}_{\text{ext}}^{i,a} = \begin{bmatrix} \mathbf{I} & \widehat{\boldsymbol{\varrho}^{i,i} - \boldsymbol{\varrho}^{i-1,i}} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \quad \text{and} \quad \mathbf{P}_{\text{ext}}^{i,b} = \begin{bmatrix} \boldsymbol{\varrho}^{i,i} \times \mathbf{n}^{i-1} \\ \mathbf{n}^{i-1} \end{bmatrix} \tag{180}$$

Recursive application of formula (179) yields

$$\mathbf{t}^i = \sum_{k=1}^{i} \mathbf{P}^{i,k} \dot{\theta}^k \tag{181}$$

with

$$\mathbf{P}^{i,k} = \begin{cases} \left(\prod_{l=i}^{k+1} \mathbf{P}_{\text{ext}}^{l,a} \right) \mathbf{P}_{\text{ext}}^{k,b} & \text{for } i > k \\ \mathbf{P}_{\text{ext}}^{k,b} & \text{for } i = k \\ \mathbf{0} & \text{for } i < k \end{cases} \tag{182}$$

for $i, k = 1, \ldots, 5$. Accordingly, the twist of the simple open kinematic chain under investigation can now be written as

$$\mathbf{t} = \mathbf{P}_{\text{ext}}^{\text{o}} \dot{\boldsymbol{\theta}} \tag{183}$$

or, more explicitly,

$$\begin{bmatrix} \mathbf{t}^1 \\ \mathbf{t}^2 \\ \mathbf{t}^3 \\ \mathbf{t}^4 \\ \mathbf{t}^5 \end{bmatrix} = \begin{bmatrix} \mathbf{P}^{1,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{P}^{2,1} & \mathbf{P}^{2,2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{P}^{3,1} & \mathbf{P}^{3,2} & \mathbf{P}^{3,3} & \mathbf{0} & \mathbf{0} \\ \mathbf{P}^{4,1} & \mathbf{P}^{4,2} & \mathbf{P}^{4,3} & \mathbf{P}^{4,4} & \mathbf{0} \\ \mathbf{P}^{5,1} & \mathbf{P}^{5,2} & \mathbf{P}^{5,3} & \mathbf{P}^{5,4} & \mathbf{P}^{5,5} \end{bmatrix} \begin{bmatrix} \dot{\theta}^1 \\ \dot{\theta}^2 \\ \dot{\theta}^3 \\ \dot{\theta}^4 \\ \dot{\theta}^5 \end{bmatrix} \tag{184}$$

Proceeding along the lines of Section 4, we arrive at the 60×5 null space matrix pertaining to the open chain

$$\mathbf{P}^{\mathrm{o}} = \mathbf{P}_{\mathrm{int}} \mathbf{P}^{\mathrm{o}}_{\mathrm{ext}} \tag{185}$$

where

$$\mathbf{P}_{\mathrm{int}} = \begin{bmatrix} \mathbf{P}^1_{\mathrm{int}} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{P}^2_{\mathrm{int}} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{P}^5_{\mathrm{int}} \end{bmatrix} \tag{186}$$

and $\mathbf{P}^i_{\mathrm{int}}$ is given by (55). It is obvious from the above treatment of the revolute pair that the discrete null space matrix pertaining to the open chain is given by

$$\mathbf{P}^{\mathrm{o}}(\mathbf{q}_n, \mathbf{q}_{n+1}) = \mathbf{P}_{\mathrm{int}}(\mathbf{q}_{n+1/2}) \mathbf{P}^{\mathrm{o}}_{\mathrm{ext}}(\mathbf{q}_{n+1/2}) \tag{187}$$

Reparametrization of unknowns: Similar to reparametrization (126) for the revolute pair, the reparametrization for the open chain is based on a product of exponentials formula which characterizes the incremental rotational motion in terms of incremental joint angles $\theta^1 \cdots \theta^5$. Accordingly, the director frame of the ith body at the end of a time step is given by

$$(\mathbf{d}^i_I)_{n+1} = \prod_{k=1}^{i} \exp(\theta^k (\widehat{\mathbf{n}^{k-1}})_n)(\mathbf{d}^i_I)_n \tag{188}$$

for $i = 1, \dots, 5$. In addition to (188), the placement of the centre of mass of the ith body with respect to G^0 (Figure 19) is given by

$$\boldsymbol{\varphi}^i_{n+1} = \sum_{l=1}^{i} ((\boldsymbol{\varrho}^{l-1,l})_{n+1} - (\boldsymbol{\varrho}^{l,l})_{n+1}) \tag{189}$$

for $i = 1, \dots, 5$.

5.4.2. Closed kinematic chain. We next consider, a closed kinematic chain consisting of six rigid bodies interconnected by revolute joints as shown in Figure 21. Our treatment of the present closed-loop system makes use of the approach outlined in Section 4.4 of Part I. To this end, we regard the open chain in Figure 19 as open-loop system associated with the closed-loop system at hand. The associated open-loop system is now subjected to additional loop-closure conditions $\widetilde{\mathbf{\Phi}}(\mathbf{q})=\mathbf{0}$. In particular, similar to (119), we get

$$\widetilde{\mathbf{\Phi}}(\mathbf{q})=\begin{bmatrix}\boldsymbol{\varphi}^5-\boldsymbol{\varphi}^0+\boldsymbol{\varrho}^{5,6}-\boldsymbol{\varrho}^{0,6}\\(\mathbf{n}^5)^{\mathrm{T}}\mathbf{d}_1^0-\eta_1^5\\(\mathbf{n}^5)^{\mathrm{T}}\mathbf{d}_2^0-\eta_2^5\end{bmatrix}\tag{190}$$

The discrete null space matrix for the closed chain can be calculated by applying the 2-step procedure outlined in Appendix C of Part I. Note that this approach essentially requires the discrete null space matrix of the associated open-loop system (187), together with reparametrizations (188), (189) and the closure conditions (190).

Numerical example: The numerical example deals with a closed kinematic chain given by the six-body linkage depicted in Figure 22. We refer to Reference [27] for a detailed investigation of the six-body linkage under consideration. There, it is shown that only four of the five loop-closure constraints (190) are independent. Thus, provided that body 0 is fixed in space, the six-body linkage has 1 DOF.

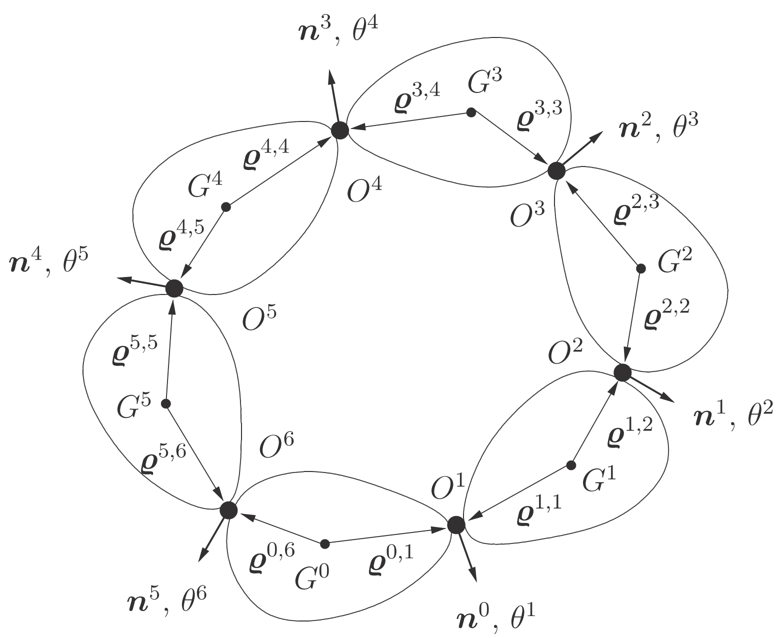

Figure 21. Schematic illustration of the closed kinematic chain.

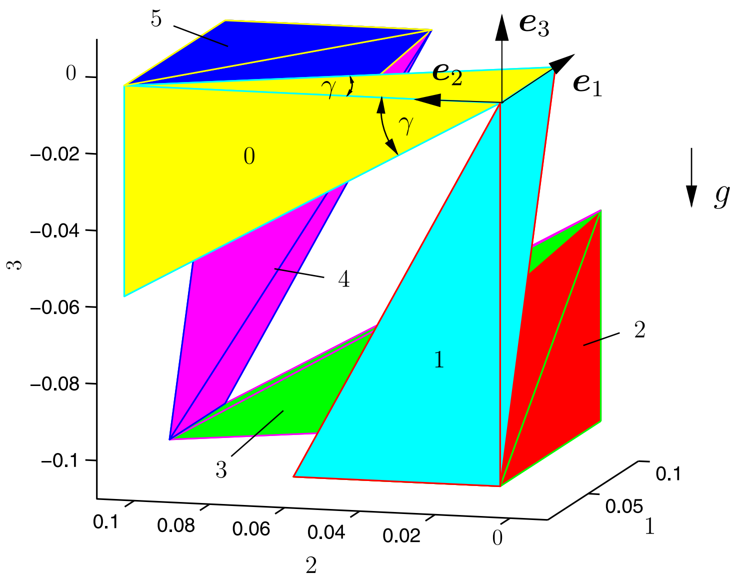

Figure 22. Initial configuration of the six-body linkage.

The initial configuration of the six-body linkage forms a cube of side length L (Figure 22). Bodies $0, 2$ and 4 are identical, and bodies $1, 3$ and 5 are identical. Furthermore, body 1 is a mirror image of body 0. Accordingly, it suffices to provide the details of body 0. Body 0 has density $\varrho = 1000$, length $L = 0.1$ and angle $\gamma = 0.16\pi$. Setting $a = \tan\gamma$, the mass of body 0 is given by $M_\varphi = \varrho a^2 L^3/6$. In the initial configuration the centre of mass of body 0 with respect to the inertial frame is given by

$$\boldsymbol{\varphi} = \varphi_i \mathbf{e}_i \quad \text{with} \quad [\varphi_i] = \begin{bmatrix} aL/4 \\ L/2 \\ -aL/4 \end{bmatrix}$$

Furthermore, the inertia tensor of body 0, with respect to its centre of mass in the initial configuration can be written as

$$\mathbf{J} = J_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$$

with

$$[J_{ij}] = \frac{\varrho L^5 a^2}{40} \begin{bmatrix} 1/3 + a^2/4 & a/6 & -a^2/12 \\ a/6 & a^2/2 & a/6 \\ -a^2/12 & a/6 & 1/3 + a^2/4 \end{bmatrix}$$

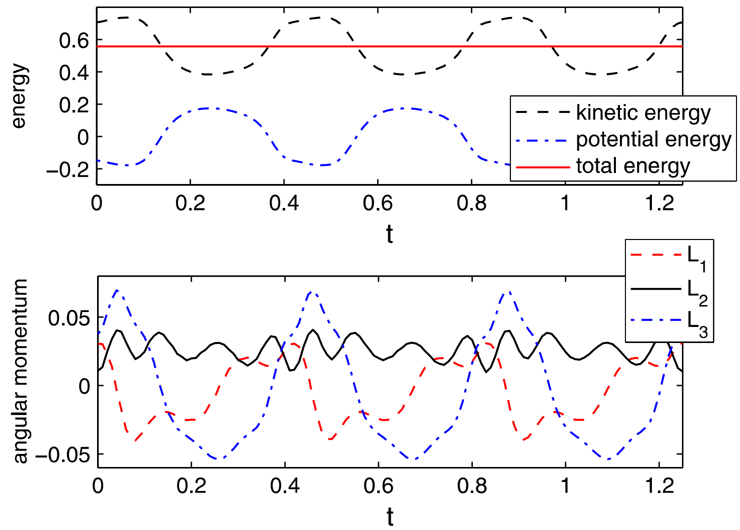

Figure 23. Six-body linkage: energy and components of angular momentum vector $\mathbf{L}=L_i\mathbf{e}_i$ ($\Delta t=0.01$).

To provide consistent initial velocities, we make use of the symmetry relationships $\theta^1=\theta^3=\theta^5$ and $\theta^2=\theta^4$, along with $\theta^2(\theta^1)=\arcsin(\sin\theta^1/(1-\sin\theta^1))$ (see Reference [27]). Thus, the generalized velocities $\dot{\boldsymbol{\theta}}\in\mathbb{R}^5$ of the associated open-loop system, see Equation (183), can be expressed as $\dot{\boldsymbol{\theta}}=\mathbf{P}^{\text{oc}}_{\text{ext}}\dot{\theta}^1$, with the 5×1 matrix

$$\mathbf{P}^{\text{oc}}_{\text{ext}}=\begin{bmatrix}1\\ \mathrm{d}\theta^2/\mathrm{d}\theta^1\\ 1\\ \mathrm{d}\theta^2/\mathrm{d}\theta^1\\ 1\end{bmatrix} \tag{191}$$

Altogether, the 60×1 null space matrix of the closed-loop system at hand can be written in the form

$$\mathbf{P}^{\text{c}}=\mathbf{P}_{\text{int}}\mathbf{P}^{\text{o}}_{\text{ext}}\mathbf{P}^{\text{oc}}_{\text{ext}} \tag{192}$$

Accordingly, consistent initial velocities $\mathbf{v}\in\mathbb{R}^{60}$ follow from $\mathbf{v}=\mathbf{P}^{\text{c}}\dot{\theta}^1$. In the numerical example $\dot{\theta}^1=30$ has been chosen. Gravity is acting on the system with $g=9.81$.

Figure 24 gives an impression of the motion by showing snapshots at consecutive instances. Algorithmic conservation of the total energy for the present conservative problem is corroborated in Figure 23. The evolution of the coordinates $\theta^1(t)$ and $\theta^2(t)$ is depicted in Figure 25.

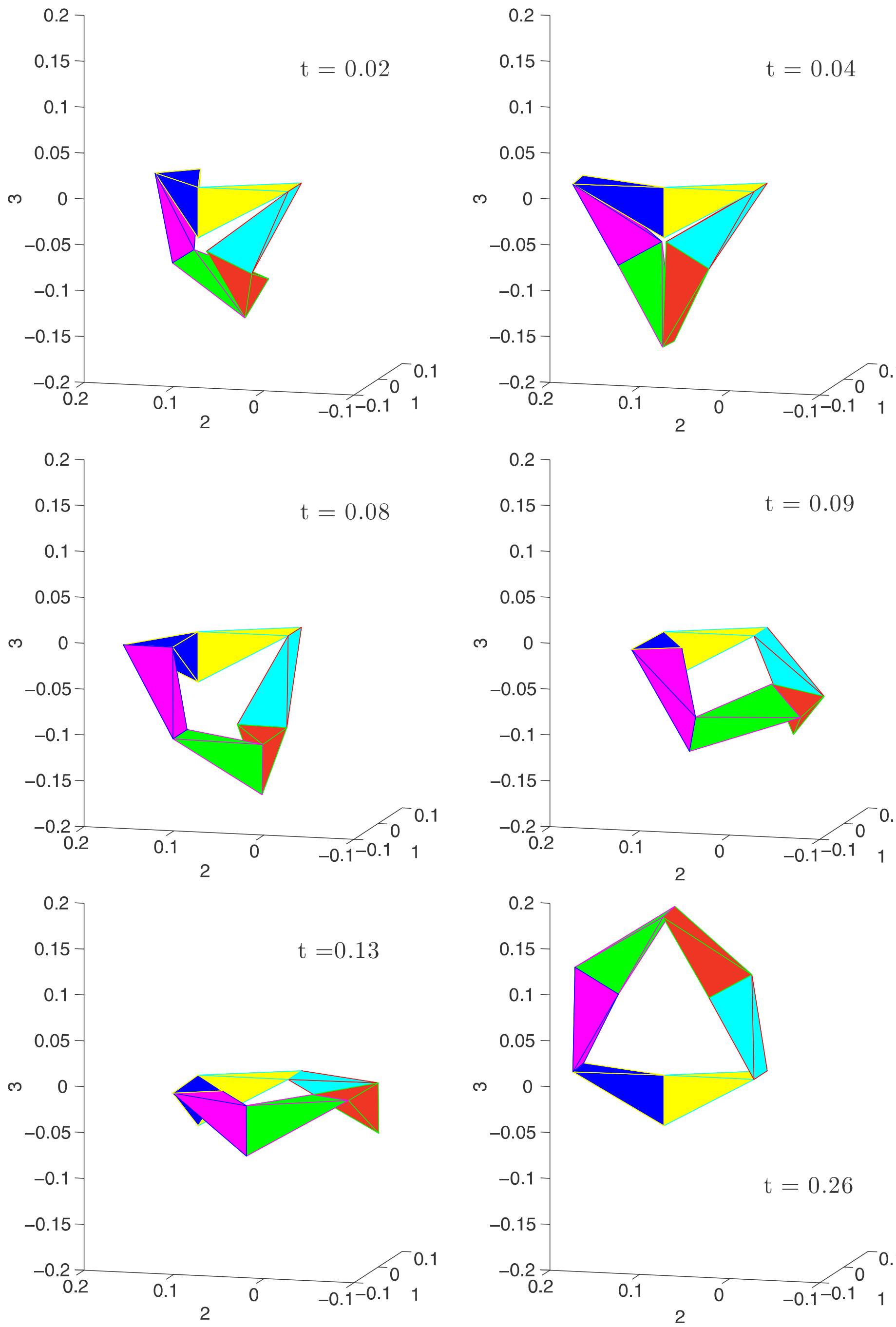

Figure 24. Six-body linkage: snapshots of the motion at $t \in \{0.02, 0.04, 0.08, 0.09, 0.13, 0.26\}$.

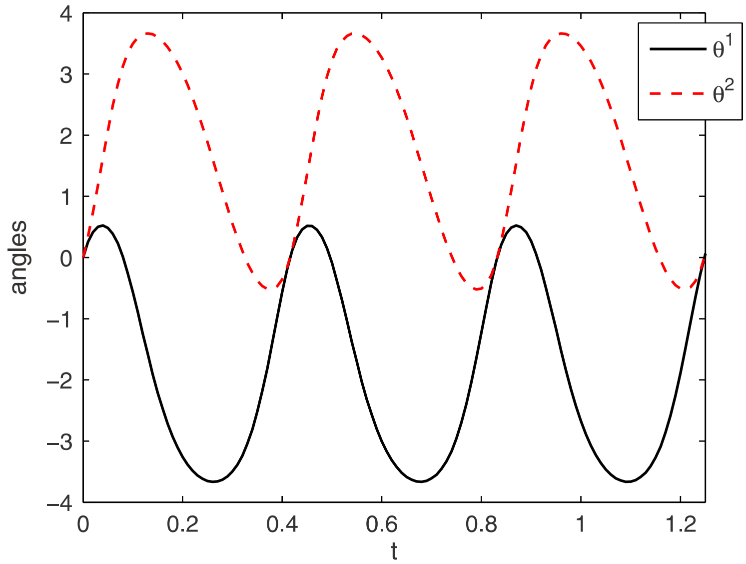

Figure 25. Six-body linkage: relative coordinates $\theta^1(t)$ and $\theta^2(t)$ ($\Delta t = 0.01$).

Table V. Comparison of the condition number of the iteration matrix for the example 'six-body linkage' ($t = 0.04$).

Δt	Constrained scheme	Reduced scheme
1×10^{-2}	$\approx 1.6 \times 10^5$	≈ 19
1×10^{-3}	$\approx 1.5 \times 10^8$	≈ 10
1×10^{-4}	$\approx 1.5 \times 10^{11}$	≈ 58

Concerning the conditioning issue, the advantageous properties of the advocated discrete null space method are obvious in view of Table V.

6. CONCLUSIONS

We have shown that the constrained formulation of rigid bodies is well-suited for the conserving discretization of multibody dynamics. In particular, the 12 redundant coordinates used for the description of each individual rigid body are supplemented with six internal constraints. In addition to that, external constraints serve the purpose of specifying a particular multibody system.

Application of the discrete null space method leads to the elimination of the constraint forces in the discrete setting, while algorithmic conservation properties and constraint fulfillment are retained. In this connection, a discrete null space matrix needs to be provided. We have shown that discrete null space matrices pertaining to specific lower kinematic pairs can be set up systematically. The extension to multibody systems can be performed in a straightforward way.

Corresponding to the division into internal and external constraints, the null space matrix pertaining to a multibody system can be decomposed in a multiplicative manner.

$$\mathbf{P} = \mathbf{P}_{\text{int}} \mathbf{P}_{\text{ext}}$$

Here $\mathbf{P}_{\text{int}}$ can be connected to the internal constraints, and always assumes the form of a block diagonal matrix, see, for example, Equations (54) or (186). On the other hand, $\mathbf{P}_{\text{ext}}$ can be linked to the external constraints, and connects the independent velocities with the twist of the system, see, for example, Equations (59) and (184).

REFERENCES

1. Betsch P. The discrete null space method for the energy consistent integration of constrained mechanical systems. Part I: Holonomic constraints. *Computer Methods in Applied Mechanics and Engineering* 2005; **194**(50–52):5159–5190.
2. José JV, Saletan EJ. *Classical Dynamics*: *A Contemporary Approach*. Cambridge University Press: Cambridge, 1998.
3. Leimkuhler B, Reich S. *Simulating Hamiltonian Dynamics*. Cambridge University Press: Cambridge, 2004.
4. Reich S. Symplectic integrators for systems of rigid bodies. *Fields Institute Communication* 1996; **10**:181–191.
5. Betsch P, Steinmann P. Constrained integration of rigid body dynamics. *Computer Methods in Applied Mechanics and Engineering* 2001; **191**:467–488.
6. Angeles J. *Fundamentals of Robotic Mechanical Systems* (2nd edn). Springer: New York, 2003.
7. Lewis D, Simo JC. Conserving algorithms for the dynamics of Hamiltonian systems on Lie groups. *Journal of Nonlinear Science* 1994; **4**:253–299.
8. Géradin M, Rixen D. Parametrization of finite rotations in computational dynamics: a review. *Revue Européenne Des Éléments Finis* 1995; **4**:497–553.
9. Krysl P. Explicit momentum-conserving integrator for dynamics of rigid bodies approximating the midpoint Lie algorithm. *International Journal for Numerical Methods in Engineering* 2005; **63**:2171–2193.
10. Chen AJ. Energy-momentum conserving methods for three dimensional dynamic nonlinear multibody systems. *Ph.D. Dissertation*, *Sudam Report No. 98-01*, Stanford University, 1998.
11. Lens EV, Cardona A, Géradin M. Energy preserving time integration for constrained multibody systems. *Multibody System Dynamics* 2004; **11**(1):41–61.
12. Ibrahimbegović A, Mamouri S, Taylor RL, Chen AJ. Finite element method in dynamics of flexible multibody systems: modeling of holonomic constraints and energy conserving integration schemes. *Multibody System Dynamics* 2000; **4**(2–3):195–223.
13. Bauchau OA, Bottasso CL. Contact conditions for cylindrical, prismatic, and screw joints in flexible multibody systems. *Multibody System Dynamics* 2001; **5**:251–278.
14. Jelenić G, Crisfield MA. Dynamic analysis of 3D beams with joints in presence of large rotations. *Computer Methods in Applied Mechanics and Engineering* 2001; **190**:4195–4230.
15. Angeles J, Lee S. The modelling of holonomic mechanical systems using a natural orthogonal complement. *Transactions of the Canadian Society for Mechanical Engineers* 1989; **13**(4):81–89.
16. Greenwood DT. *Principles of Dynamics*. Prentice-Hall, Englewood Cliffs, NJ, 1988.
17. Simo JC, Wong KK. Unconditionally stable algorithms for rigid body dynamics that exactly preserve energy and momentum. *International Journal for Numerical Methods in Engineering* 1991; **31**:19–52.
18. Bottasso CL, Borri M. Integrating finite rotations. *Computer Methods in Applied Mechanics and Engineering* 1998; **164**:307–331.
19. Haug EJ. *Computer-Aided Kinematics and Dynamics of Mechanical Systems. Volume I*: *Basic Methods*. Allyn & Bacon: Newton, MA, 1989.
20. Garcia de Jalon J, Bayo E. *Kinematic and Dynamic Simulation of Multibody Systems*: *The Real-Time Challenge*. Springer: New York, 1994.
21. Taylor RL. Finite element analysis of rigid-flexible systems. In *Computational Aspects of Nonlinear Structural Systems with Large Rigid Body Motion*, Ambrósio JAC, Kleiber M (eds), NATO Science Series: Computer & Systems Sciences, vol. 179, IOS Press: Bristol, 2001; 63–84.

22. Betsch P, Steinmann P. A DAE approach to flexible multibody dynamics. *Multibody System Dynamics* 2002; **8**:367–391.
23. Angeles J. *Rational Kinematics*. Springer: New York, 1988.
24. Marsden JE, Ratiu TS. *Introduction to Mechanics and Symmetry* (2nd edn). Springer: New York, 1999.
25. Géradin M, Cardona A. *Flexible Multibody Dynamics*: *A Finite Element Approach*. Wiley: New York, 2001.
26. Moon FC. *Applied Dynamics*. Wiley: New York, 1998.
27. Wittenburg J. *Dynamics of Systems of Rigid Bodies*. Teubner-Verlag: Stuttgart, 1977.

